Automated Proofs of Unique Normal Forms w.r.t. Conversion for Term Rewriting Systems

Takahito Aoto[1] and Yoshihito Toyama[2]

[1] School of Natural Sciences, Niigata University, JAPAN
aoto@ie.niigata-u.ac.jp
[2] RIEC, Tohoku University, JAPAN
toyama@riec.tohoku.ac.jp

Abstract. The notion of normal forms is ubiquitous in various equivalent transformations. Confluence (CR), one of the central properties of term rewriting systems (TRSs), concerns uniqueness of normal forms. Yet another such property, which is weaker than confluence, is the property of unique normal forms w.r.t. conversion (UNC). Recently, automated confluence proof of TRSs has caught attentions; some powerful confluence tools integrating multiple methods for (dis)proving the CR property of TRSs have been developed. In contrast, there have been little efforts on (dis)proving the UNC property automatically yet. In this paper, we report on a UNC prover combining several methods for (dis)proving the UNC property. We present an equivalent transformation of TRSs preserving UNC, as well as some new criteria for (dis)proving UNC.

1 Introduction

The notion of normal forms is ubiquitous in various equivalent transformations—normal forms are objects that cannot be transformed further. A crucial issue around the notion of normal forms is that whether they are unique so that normal forms (if exist) can represent the equivalence classes of objects. For this, the notion of confluence (CR), namely that $s \xleftarrow{*} \circ \xrightarrow{*} t$ implies $s \xrightarrow{*} \circ \xleftarrow{*} t$ for all objects s and t, is most well-studied. Here, $\xrightarrow{*}$ is the reflexive transitive closure of the equivalent transformation $\to$, and $\circ$ stands for the composition. In term rewriting, various methods for proving confluence of term rewriting systems (TRSs) have been studied (see e.g. slides of [20] for a survey). Yet another such a property is the property of unique normal forms w.r.t. conversion (UNC)[3], namely that two convertible normal forms are identical, i.e. $s \xleftrightarrow{*} t$ with normal forms s, t implies $s = t$. In term rewriting, famous examples that are UNC but not CR include TRSs consisting of S,K,I-rules for combinatory logic supplemented with various pairing rules [13, 22], whose non-CR have been shown in [12].

[3] The uniqueness of normal forms w.r.t. conversion is also often abbreviated as UN in the literature; here, we prefer UNC to distinguish it from a similar but different notion of unique normal forms w.r.t. reduction (UNR), following the convention employed in CoCo (Confluence Competition).

It is undecidable whether $\mathcal{R}$ is UNC for a given TRS $\mathcal{R}$ in general. However, it is known that the UNC property is decidable for left-linear right-ground TRSs [6] and for shallow TRSs [17]. Another class for which the UNC property is decidable is terminating TRSs, for which the CR property and the UNC property coincide (e.g. [7]). Some classes of TRSs having the UNC property are also known: non-ω-overlapping TRSs [10] and non-duplicating weight-decreasing joinable TRSs [21]. Another important topic on the UNC property is modularity. It is known that the UNC property is modular for persistent decomposition [2] and layer-preserving decomposition [1]. These results allow us to use the divide-and-conquer approach for (dis)proving the UNC property. Compared to the CR property, however, not much has been studied on the UNC property in the field of term rewriting.

Recently, automated confluence proof of TRSs has caught attentions leading to investigations of automatable methods for (dis)proving the CR property of TRSs; some powerful confluence tools have been developed as well, such as ACP [3], CSI [14], Saigawa [11] for TRSs, and also tools for other frameworks such as conditional TRSs and higher-order TRSs. This leads to the emergence of the Confluence Competition (CoCo)[4], yearly efforts since 2012. In contrast, there have been little efforts on (dis)proving the UNC property automatically. Indeed, there are few tools that are capable of (dis)proving the UNC property; furthermore, only few UNC criteria have been elaborated in these tools.

In this paper, we report on a UNC prover comprising multiple methods for (dis)proving the UNC property and integrating them in a modular way. We present new automated methods to prove or disprove the UNC property; these methods enabled our tool to win the UNC category of CoCo 2018.

The rest of the paper is organized as follows. After introducing necessary notions and notations in Section 2, we first revisit the conditional linearization technique for proving UNC, and obtain new UNC criteria based on this approach in Section 3. In Section 4, we present a slightly generalized version of the critical pair criterion presented in the paper [21], and report an automation of the criterion. In Section 5, we present a new method for proving or disproving UNC. We show an experiment of the presented methods in Section 6. In Section 7, we report our prover ACP which supports the presented methods and integrates them based on the modularity results. Section 8 concludes.

2 Preliminaries

We now fix notions and notations used in the paper. We assume familiarity with basic notions in term rewriting (e.g. [4]).

We use $\sqcup$ to denote the multiset union and $\mathbb{N}$ the set of natural numbers. A sequence of objects $a_1, \ldots, a_n$ is written as $\boldsymbol{a}$. Negation of a predicate P is denoted by $\neg P$. The composition of relation R and S is denoted by $R \circ S$. Let $\to$ be a relation on a set A. The reflexive transitive (reflexive, symmetric, equivalent) closure of the relation $\to$ is denoted by $\xrightarrow{*}$ (resp. $\xrightarrow{=}$, $\leftrightarrow$, $\xleftrightarrow{*}$). The set

[4] http://project-coco.uibk.ac.at/

NF of *normal forms* w.r.t. the relation $\to$ is given by $\mathrm{NF} = \{a \in A \mid a \to b$ for no $b \in A\}$. The relation $\to$ has *unique normal forms w.r.t. conversion* (denoted by $\mathrm{UNC}(\to)$) if $a \overset{*}{\leftrightarrow} b$ and $a, b \in \mathrm{NF}$ imply $a = b$. The relation $\to$ is *confluent* (denoted by $\mathrm{CR}(\to)$) if $\overset{*}{\leftarrow} \circ \overset{*}{\to} \subseteq \overset{*}{\to} \circ \overset{*}{\leftarrow}$. When we consider two relations $\to_1$ and $\to_2$, the respective sets of normal forms w.r.t. $\to_1$ and $\to_2$ are denoted by NF_1 and NF_2. The following proposition, which is proved easily, is a basis of the *conditional linearization technique*, which will be used in Sections 3 and 4.

Proposition 1 ([13, 22]). *Suppose (1)* $\to_0 \subseteq \to_1$*, (2)* $\mathrm{CR}(\to_1)$*, and (3)* $\mathrm{NF}_0 \subseteq \mathrm{NF}_1$*. Then,* $\mathrm{UNC}(\to_0)$*.*

The set of *terms* over the set $\mathcal{F}$ of fixed-arity *function symbols* and denumerable set $\mathcal{V}$ of *variables* is denoted by $\mathrm{T}(\mathcal{F}, \mathcal{V})$. The set of variables in a term t is denoted by $\mathcal{V}(t)$. A term t is *ground* if $\mathcal{V}(t) = \emptyset$. We abuse the notation $\mathcal{V}(t)$ and denote by $\mathcal{V}(e)$ the set of variables occurring in any sequence e of expressions. The *subterm* of a term t at a *position* p is denoted by $t|_p$. The root position is denoted by ϵ. A *context* is a term containing a special constant $\Box$ (called *hole*). If C is a context containing n-occurrences of the hole, $C[t_1, \ldots, t_n]_{p_1,\ldots,p_n}$ denotes the term obtained from C by replacing holes with $t_1, \ldots, t_n$ at the positions $p_1, \ldots, p_n$. Here, subscripts $p_1, \ldots, p_n$ may be abbreviated if it can be remained implicit. The expression $s[t_1, \ldots, t_n]_{p_1,\ldots,p_n}$ denotes the term obtained from s by replacing subterms at the positions $p_1, \ldots, p_n$ with terms $t_1, \ldots, t_n$ respectively. We denote by $|t|_x$ the number of occurrences of a variable x in a term t. Again, we abuse the notation $|t|_x$ and denote by $|e|_x$ the number of occurrences of a variable x in any sequence of expressions e. A term t is *linear* if $|t|_x = 1$ for any $x \in \mathcal{V}(t)$. A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathrm{T}(\mathcal{F}, \mathcal{V})$ with finite $\mathrm{dom}(\sigma) = \{x \in \mathcal{V} \mid \sigma(x) \neq x\}$. Each substitution is identified with its homomorphic extension over $\mathrm{T}(\mathcal{F}, \mathcal{V})$. For simplicity, we often write $t\sigma$ instead of $\sigma(t)$. A *most general unifier* σ of terms s and t is denoted by $\mathrm{mgu}(s, t)$.

An *equation* is a pair $\langle l, r\rangle$ of terms, which is denoted by $l \approx r$. When we do not distinguish the lhs and rhs of the equation, we write $l \stackrel{.}{\approx} r$. We identify equations modulo renaming of variables. For a set or sequence Γ of equations, we denote by $\Gamma\sigma$ the set or the sequence obtained by replacing each equation $l \approx r$ by $l\sigma \approx r\sigma$. An equation $l \approx r$ satisfying $l \notin \mathcal{V}$ and $\mathcal{V}(r) \subseteq \mathcal{V}(l)$ is a *rewrite rule* and written as $l \to r$. A rewrite rule $l \to r$ is *linear* if l and r are linear terms; it is *left-linear* (*right-linear*) if l (resp. r) is a linear term. A rewrite rule $l \to r$ is *non-duplicating* if $|l|_x \geq |r|_x$ for any $x \in \mathcal{V}(l)$. A *term rewriting system* (*TRS*, for short) is a finite set of rewrite rules. A TRS is linear (left-linear, right-linear, non-duplicating) if so are all rewrite rules. A *rewrite step* of a TRS $\mathcal{R}$ (a set Γ of equations) is a relation $\to_{\mathcal{R}}$ (resp. $\leftrightarrow_{\Gamma}$) over $\mathrm{T}(\mathcal{F}, \mathcal{V})$ defined by $s \to_{\mathcal{R}} t$ iff $s = C[l\sigma]$ and $t = C[r\sigma]$ for some $l \to r \in \mathcal{R}$ (resp. $l \stackrel{.}{\approx} r \in \Gamma$) and context C and substitution σ. The position p such that $C|_p = \Box$ is called the *redex position* of the rewrite step, and we write $s \to_{p,\mathcal{R}} t$ to indicate the redex position explicitly. A *rewrite sequence* is (finite or infinite) consecutive applications of rewrite steps. A rewrite sequence of the form $t_1 \;{}_{\mathcal{R}}{\leftarrow}\; t_0 \to_{\mathcal{R}} t_2$ is called a *local peak*.

Let $l_1 \to r_1$ and $l_2 \to r_2$ be rewrite rules such that $\mathcal{V}(l_1) \cap \mathcal{V}(l_2) = \emptyset$. Let $\sigma = \mathrm{mgu}(l_1, l_2|_p)$ with $l_2|_p \notin \mathcal{V}$. A local peak $l_2[r_1]_p\sigma \;{}_{p,\mathcal{R}}{\leftarrow}\; l_2\sigma \to_{\epsilon,\mathcal{R}} r_2\sigma$ is called a *critical peak* of $l_1 \to r_1$ over $l_2 \to r_2$, provided that $p \neq \epsilon$ or $(l_1 \to r_1) \neq (l_2 \to r_2)$. The pair $\langle l_2[r_1]_p\sigma, r_2\sigma\rangle$ is called a *critical pair* in $\mathcal{R}$. It is *overlay* if $p = \epsilon$; it is *inner-outer* if $p \neq \epsilon$. The set of (overlay, inner-outer) critical pairs from rules in a TRS $\mathcal{R}$ is denoted by $\mathrm{CP}(\mathcal{R})$ (resp. $\mathrm{CP}_{out}(\mathcal{R})$, $\mathrm{CP}_{in}(\mathcal{R})$).

Let $l \approx r$ be an equation and let Γ be a finite sequence of equations. An expression of the form $\Gamma \Rightarrow l \approx r$ is called a *conditional equation*. Conditional equations are also identified modulo renaming of variables. If $l \notin \mathcal{V}$, it is a *conditional rewrite rule* and written as $l \to r \Leftarrow \Gamma$. The sequence Γ is called the *condition part* of the rule.

A conditional rewrite rule $l \to r \Leftarrow \Gamma$ is linear (left-linear) if so are rewrite rule $l \to r$. A finite set of conditional rewrite rules is called a *conditional term rewriting system* (*CTRS*, for short). A CTRS is linear (left-linear) if so are all rules. A CTRS $\mathcal{R}$ is said to be of *type 1* if $\mathcal{V}(\Gamma) \cup \mathcal{V}(r) \subseteq \mathcal{V}(l)$ for all $l \to r \Leftarrow \Gamma \in \mathcal{R}$.

Let $l_1 \to r_1 \Leftarrow \Gamma_1$ and $l_2 \to r_2 \Leftarrow \Gamma_2$ be conditional rewrite rules such that w.l.o.g. $\mathcal{V}(l_1, r_1, \Gamma_1) \cap \mathcal{V}(l_2, r_2, \Gamma_2) = \emptyset$. Let $\sigma = \mathrm{mgu}(l_1, l_2|_p)$ with $l_2|_p \notin \mathcal{V}$. Then $\Gamma_1\sigma, \Gamma_2\sigma \Rightarrow \langle l_2[r_1]_p\sigma, r_2\sigma\rangle$ is called a *conditional critical pair* (CCP, for short), provided that $p \neq \epsilon$ or $(l_1 \to r_1 \Leftarrow \Gamma_1) \neq (l_2 \to r_2 \Leftarrow \Gamma_2)$. Here, $\Gamma_1\sigma, \Gamma_2\sigma$ is the juxtaposition of sequences $\Gamma_1\sigma$ and $\Gamma_2\sigma$. It is overlay if $p = \epsilon$; it is inner-outer if $p \neq \epsilon$. The set of (overlay, inner-outer) CCPs from rules in a CTRS $\mathcal{R}$ is denoted by $\mathrm{CCP}(\mathcal{R})$ (resp. $\mathrm{CCP}_{out}(\mathcal{R})$, $\mathrm{CCP}_{in}(\mathcal{R})$). A CTRS $\mathcal{R}$ is *orthogonal* if it is left-linear and $\mathrm{CCP}(\mathcal{R}) = \emptyset$.

In this paper, we deal with *semi-equational* CTRSs. The conditional rewrite step $\to_{\mathcal{R}} = \bigcup_{n \in \mathbb{N}} \to_{\mathcal{R}}^{(n)}$ of a semi-equational CTRS $\mathcal{R}$ is given via auxiliary relations $\to_{\mathcal{R}}^{(n)}$ $(n \geq 0)$ defined like this: $\to_{\mathcal{R}}^{(0)} = \emptyset$, $\to_{\mathcal{R}}^{(n+1)} = \{\langle C[l\sigma], C[r\sigma]\rangle \mid l \to r \Leftarrow s_1 \approx t_1, \ldots, s_k \approx t_k \in \mathcal{R}, \forall i\ (1 \leq i \leq k).\ s_i\sigma \overset{*}{\leftrightarrow}{}_{\mathcal{R}}^{(n)} t_i\sigma)\}$. The *rank* of a conditional rewrite step $s \to_{\mathcal{R}} t$ is the least n such that $s \to_{\mathcal{R}}^{(n)} t$.

Let $\mathcal{R}$ be a TRS or CTRS. The set of normal forms w.r.t. $\to_{\mathcal{R}}$ is written as $\mathrm{NF}(\mathcal{R})$. A (C)TRS $\mathcal{R}$ is UNC (CR) if $\mathrm{UNC}(\to_{\mathcal{R}})$ (resp. $\mathrm{CR}(\to_{\mathcal{R}})$) on the set $\mathrm{T}(\mathcal{F}, \mathcal{V})$. Let $\mathcal{E}$ be a set or sequence of equations or rewrite rules. We denote $\approx_{\mathcal{E}}$ the congruence closure of $\mathcal{E}$. We write $\vdash_{\mathcal{E}} l \approx r$ if $l \overset{*}{\leftrightarrow}_{\mathcal{E}} r$. For sets or sequences Γ and Σ of equations, we write $\vdash_{\mathcal{E}} \Sigma$ if $\vdash_{\mathcal{E}} l \approx r$ for all $l \approx r \in \Sigma$, and $\Gamma \vdash_{\mathcal{E}} \Sigma$ if $\vdash_{\mathcal{E}} \Gamma\sigma$ implies $\vdash_{\mathcal{E}} \Sigma\sigma$ for any substitution σ.

A TRS $\mathcal{R}$ is said to be *right-reducible* if $r \notin \mathrm{NF}(\mathcal{R})$ for all $l \to r \in \mathcal{R}$. Although it is straightforward, we did not noticed the following claim having appeared in the literature:

Proposition 2. *Right-reducible TRSs are* UNC*.*

Example 1 (Cops ♯126). Let $\mathcal{R} = \{f(f(x, y), z) \to f(f(x, z), f(y, z))\}$. The state of the art confluence tools fail to prove confluence of this example. However, it is easy to see $\mathcal{R}$ is right-reducible, and thus, the UNC property is easily obtained automatically.

3 Conditional linearization revisited

In this section, we revisit the conditional linearization technique.

3.1 Conditional linearization

A conditional linearization is a translation from TRSs to CTRSs which eliminates non-left-linear rewrite rules, say $f(x, x) \to r$, by replacing them with a corresponding conditional rewrite rules, such as $f(x, y) \to r \Leftarrow x \approx y$. Formally, let $l = C[x_1, \dots, x_n]$ with all variable occurrences in l displayed (i.e. $\mathcal{V}(C) = \emptyset$). Note here l may be a non-linear term and some variables in $x_1, \dots, x_n$ may be identical. Let $l' = C[x'_1, \dots, x'_n]$ where $x'_1, \dots, x'_n$ are mutually distinct fresh variables and let δ be a substitution such that $\delta(x'_i) = x_i$ $(1 \leq i \leq n)$ and $\mathrm{dom}(\delta) = \{x'_1, \dots, x'_n\}$. A conditional rewrite rule $l' \to r' \Leftarrow \Gamma$ is a *conditional linearization* of a rewrite rule $l \to r$ if $r'\delta = r$ and Γ is a sequence of equations $x_i \approx x_j$ $(1 \leq i, j \leq n)$ such that $x'_i \approx_\Gamma x'_j$ iff $x'_i\delta = x'_j\delta$ holds for all $1 \leq i, j \leq n$, where $\approx_\Gamma$ is the congruence closure of Γ. A conditional linearization of a TRS $\mathcal{R}$ is a semi-equational CTRS (denoted by $\mathcal{R}^L$) obtained by replacing each rewrite rule with its conditional linearization. We remark that any result of conditional linearization is a left-linear CTRS of type 1.

Conditional linearization is useful for showing the UNC property of non-left-linear TRSs. The key observation is $\mathrm{CR}(\mathcal{R}^L)$ implies $\mathrm{UNC}(\mathcal{R})$. For this, we use Proposition 1 for $\to_0 := \to_\mathcal{R}$ and $\to_1 := \to_{\mathcal{R}^L}$. Clearly, $\to_\mathcal{R} \subseteq \to_{\mathcal{R}^L}$, and thus the condition (1) of Proposition 1 holds. Suppose $\mathrm{CR}(\mathcal{R}^L)$. Then, one can easily show that $\mathrm{NF}(\mathcal{R}) \subseteq \mathrm{NF}(\mathcal{R}^L)$ by induction on the rank of conditional rewrite steps. Thus, the condition (2) of Proposition 1 implies its condition (3). Hence, $\mathrm{CR}(\mathcal{R}^L)$ implies $\mathrm{UNC}(\mathcal{R})$.

Now, for semi-equational CTRSs, the following confluence criterion is known.

Proposition 3 ([5, 15]). *Orthogonal semi-equational CTRSs are confluent.*

A TRS $\mathcal{R}$ is *strongly non-overlapping* if $\mathrm{CCP}(\mathcal{R}^L) = \emptyset$. Hence, it follows:

Proposition 4 ([13, 22]). *Strongly non-overlapping TRSs are UNC.*

This proposition is subsumed by the UNC of non-ω-overlapping TRSs [10].

3.2 UNC by conditional linearization

We now give some simple extensions of Proposition 4 which are easily incorporated from [8], but does not fall within the class of non-ω-overlapping TRSs. For this, let us recall the notion of parallel rewrite steps. A *parallel rewrite step* $s \mathrel{+\!\!\!\!\!\!\longrightarrow}_\mathcal{R} t$ is defined like this: $s \mathrel{+\!\!\!\!\!\!\longrightarrow}_\mathcal{R} t$ iff $s = C[l_1\sigma_1, \dots, l_n\sigma_n]$ and $t = C[r_1\sigma_1, \dots, r_n\sigma_n]$ for some rewrite rules $l_1 \to r_1, \dots, l_n \to r_n \in \mathcal{R}$ and context C and substitutions $\sigma_1, \dots, \sigma_n$ $(n \geq 0)$. Let us write $\Gamma \vdash_\mathcal{R} u \to v$ if $\vdash_\mathcal{R} \Gamma\sigma$ implies $u\sigma \to_\mathcal{R} v\sigma$ for any substitution σ. We define $\Gamma \vdash_\mathcal{R} u \mathrel{+\!\!\!\!\!\!\longrightarrow}_\mathcal{R} v$, etc. analogously.

The following notion is a straightforward extension of the corresponding notion in [8, 19].

Definition 1. *A semi-equational CTRS $\mathcal{R}$ is* parallel-closed *if (i) $\Gamma \vdash_{\mathcal{R}} u \mathrel{+\!\!\!\!\!\!\longrightarrow} v$ for any inner-outer CCP $\Gamma \Rightarrow \langle u, v \rangle$ of $\mathcal{R}$, and (ii) $\Gamma \vdash_{\mathcal{R}} u \mathrel{+\!\!\!\!\!\!\longrightarrow} \circ \xleftarrow{*} v$ for any overlay CCP $\Gamma \Rightarrow \langle u, v \rangle$ of $\mathcal{R}$.*

We now come to our first extension of Proposition 4, which is proved in a way very similar to the one for TRSs.

Theorem 1. *Parallel-closed semi-equational CTRSs of type 1 are confluent.*

Corollary 1. *A TRS $\mathcal{R}$ is UNC if $\mathcal{R}^L$ is parallel-closed.*

Example 2. Let $\mathcal{R} = \{\ @(@(@(S,x),y),z) \to @(@(x,z),@(y,z)),\ @(@(K,x),y) \to x,\ @(I,x) \to x,\ @(@(D,x),x) \to x,\ \mathsf{app}(K,x) \to @(I,x),\ \mathsf{app}(x,K) \to x\ \}$. Since $\mathcal{R}$ is non-terminating, non-shallow, and non-right-ground, previous decidability results for UNC does not apply. Furthermore, since $\mathcal{R}$ is overlapping and duplicating, previous sufficient criteria for UNC does not apply. Also, previous modularity results for UNC does not properly decompose $\mathcal{R}$. Note that the TRS consisting of the first 4 rules is a famous non-confluent example ([12]); one can prove that $\mathcal{R}$ is non-confluent in a similar way. We have $\mathrm{CCP}_{in}(\mathcal{R}^L) = \emptyset$ and $\mathrm{CCP}_{out}(\mathcal{R}^L) = \{\emptyset \Rightarrow \langle @(I,K), K \rangle, \emptyset \Rightarrow \langle K, @(I,K) \rangle\}$. Thus, $\mathcal{R}^L$ is parallel-closed, and from Corollary 1, it follows that $\mathcal{R}$ is UNC.

Next, we incorporate the strong confluence criterion of TRSs [8] to semi-equational CTRSs in the similar way.

Definition 2. *A semi-equational CTRS $\mathcal{R}$ is* strongly closed *if $\Gamma \vdash_{\mathcal{R}} u \xrightarrow{*} \circ \xleftarrow{=} v$ and $\Gamma \vdash_{\mathcal{R}} u \xrightarrow{=} \circ \xleftarrow{*} v$ for any CCP $\Gamma \Rightarrow \langle u, v \rangle$ of $\mathcal{R}$.*

Similar to the proof of Theorem 1, the following theorem is obtained in the same way as in the proof for TRSs.

Theorem 2. *Linear strongly closed semi-equational CTRSs of type 1 are confluent.*

Corollary 2. *A TRS $\mathcal{R}$ is UNC if $\mathcal{R}^L$ is linear and strongly closed.*

We remark that the results of conditional linearization are not unique. Although the rewrite relation $\to_{\mathcal{R}^L}$ is independent of the results of conditional linearization, the CCPs may be different depending on $\mathcal{R}^L$. Thus, the applicability of Theorems 1 and 2 changes by the choice of $\mathcal{R}^L$. This is exhibited in the next example, where the first 5 rules are from [8].

Example 3. Let

$$\mathcal{R} = \left\{ \begin{array}{lll} H(F(x,y)) \to F(H(R(x)),y) & F(x,K(y,z)) & \to G(P(y),Q(z,x)) \\ H(Q(x,y)) \to Q(x,H(R(y))) & Q(x,H(R(y))) & \to H(Q(x,y)) \\ H(G(x,y)) \to G(x,H(y)) & K(x,x) & \to R(x) \\ P(y) \qquad\quad\ \to C & C \to K(C,C) & \\ F(x,R(y)) \ \to G(C,Q(y,x)) & & \end{array} \right\}.$$

There are two variants of conditional linearization of the sixth rule, namely $K(x_1, x_2) \to R(x_1) \Leftarrow x_1 \approx x_2$ and $K(x_1, x_2) \to R(x_2) \Leftarrow x_1 \approx x_2$. Depending on the choice of the variants, one obtains two kinds of CCP—namely, $\langle F(x, R(y)), G(P(y), Q(z, x))\rangle$ and $\langle F(x, R(z)), G(P(y), Q(z, x))\rangle$. The former is strongly closed as $F(x, R(y)) \to G(C, Q(y, x) \leftarrow G(P(y), Q(z, x))$. On the other hand, the latter is not. Actually, the CTRS obtained by the former linearization is strongly closed, while the CTRS obtained by the latter linearization is not strongly closed.

3.3 Automation

Even though proofs are rather straightforward, it is not at all obvious how the conditions of Theorems 1 and 2 can be effectively checked.

Let $\mathcal{R}$ be a semi-equational CTRS. Let $\Gamma \Rightarrow \langle u, v\rangle$ be an inner-outer CCP of $\mathcal{R}$, and consider to check $\Gamma \vdash_{\mathcal{R}} u \mathrel{-\!\!\!\mapsto\!\!\!\!\!\to} v$. For this, we construct the set $\mathit{Red} = \{v' \mid \Gamma \vdash_{\mathcal{R}} u \mathrel{-\!\!\!\mapsto\!\!\!\!\!\to} v'\}$ and check whether $v \in \mathit{Red}$. To construct the set Red, we seek the possible redex positions in u. Suppose we found conditional rewrite rules $l_1 \to r_1 \Leftarrow \Gamma_1, l_2 \to r_2 \Leftarrow \Gamma_2 \in \mathcal{R}$ and substitutions θ_1, θ_2 such that $u = C[l_1\theta_1, l_2\theta_2]$. Then we obtain $u \mathrel{-\!\!\!\mapsto\!\!\!\!\!\to} C[r_1\theta_1, r_2\theta_2]$ if $\vdash_{\mathcal{R}} \Gamma_1\theta_1$ and $\vdash_{\mathcal{R}} \Gamma_2\theta_2$, i.e. $s \stackrel{*}{\leftrightarrow}_{\mathcal{R}} t$ for any equation $s \approx t$ in $\Gamma_1\theta_1 \cup \Gamma_2\theta_2$. Now, for checking $\Gamma \vdash_{\mathcal{R}} u \mathrel{-\!\!\!\mapsto\!\!\!\!\!\to} v$, it suffices to consider the case $\vdash_{\mathcal{R}} \Gamma$ holds. Thus, we may assume $s' \stackrel{*}{\leftrightarrow}_{\mathcal{R}} t'$ for any $s' \approx t'$ in Γ. Therefore, the problem is to check whether $s' \stackrel{*}{\leftrightarrow}_{\mathcal{R}} t'$ for $s' \approx t'$ in Γ implies $s \stackrel{*}{\leftrightarrow}_{\mathcal{R}} t$ for any equation $s \approx t$ in $\Gamma_1\theta_1 \cup \Gamma_2\theta_2$.

To check this, we use the following sufficient condition: $s \approx_\Gamma t$ for all $s \approx t \in \Gamma_1\theta_1 \cup \Gamma_2\theta_2$. Since congruence closure of a finite set of equations is recursive (e.g. [4]), this approximation is indeed automatable.

Example 4. Let

$$\mathcal{R} = \left\{ \begin{array}{lll} P(Q(x)) \to P(R(x)) \Leftarrow x \approx A & & Q(H(x)) \to R(x) \Leftarrow S(x) \approx H(x) \\ R(x) \quad\;\; \to R(H(x)) \Leftarrow S(x) \approx A & & \end{array} \right\}.$$

Then, we have $\mathrm{CCP}(\mathcal{R}) = \mathrm{CCP}_{in}(\mathcal{R}) = \{S(x) \approx H(x), H(x) \approx A \Rightarrow \langle P(R(x)), P(R(H(x)))\rangle\}$. In order to apply the third rule to have $P(R(x)) \mathrel{-\!\!\!\mapsto\!\!\!\!\!\to}_{\mathcal{R}} P(R(H(x)))$, we have to check the condition $S(x) \stackrel{*}{\leftrightarrow}_{\mathcal{R}} A$. This holds, since we can suppose $S(x) \stackrel{*}{\leftrightarrow}_{\mathcal{R}} H(x)$ and $H(x) \stackrel{*}{\leftrightarrow}_{\mathcal{R}} A$. This is checked by $S(x) \approx_\Sigma A$, where $\Sigma = \{S(x) \approx H(x), H(x) \approx A\}$.

4 Automating UNC proof of non-duplicating TRSs

In this section, we show a slight generalization of the UNC criterion of the paper [21], and show how the criterion can be decided. First, we briefly capture necessary notions and notations from the paper [21].

A *left-right separated (LR-separated) conditional rewrite rule* is $l \to r \Leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n$ such that (i) $l \notin \mathcal{V}$ is linear, (ii) $\mathcal{V}(l) = \{x_1, \ldots, x_n\}$ and

$$\frac{}{\Gamma \sqcup \{u \approx v\} \Vdash_{\mathcal{R}} u \sim_0 v} \qquad \frac{}{\Gamma \Vdash_{\mathcal{R}} t \sim_0 t} \qquad \frac{\Gamma \Vdash_{\mathcal{R}} t \sim_i s}{\Gamma \Vdash_{\mathcal{R}} s \sim_i t} \qquad \frac{\Gamma \Vdash_{\mathcal{R}} s \sim_i t \quad \Sigma \Vdash_{\mathcal{R}} t \sim_j u}{\Gamma \sqcup \Sigma \Vdash_{\mathcal{R}} s \sim_{i+j} u}$$

$$\frac{\Gamma \Vdash_{\mathcal{R}} s \sim_i t}{\Gamma \Vdash_{\mathcal{R}} C[s] \sim_i C[t]} \qquad \frac{\Gamma_1 \Vdash_{\mathcal{R}} u_1 \sim_{i_1} v_1 \quad \cdots \quad \Gamma_n \Vdash_{\mathcal{R}} u_n \sim_{i_n} v_n}{\bigsqcup_j \Gamma_j \Vdash_{\mathcal{R}} \langle u_1, \ldots, u_n \rangle \sim_k \langle v_1, \ldots, v_n \rangle} \; k = \sum_j i_j$$

$$\frac{\Gamma \Vdash_{\mathcal{R}} s \to_i t}{\Gamma \Vdash_{\mathcal{R}} s \sim_i t} \qquad \frac{\Gamma \Vdash_{\mathcal{R}} \langle u_1\sigma, \ldots, u_n\sigma \rangle \sim_i \langle v_1\sigma, \ldots, v_n\sigma \rangle}{\Gamma \Vdash_{\mathcal{R}} C[l\sigma] \to_{i+1} C[r\sigma]} \; l \to r \Leftarrow u_1 \approx v_1, \ldots, u_n \approx v_n \in \mathcal{R}$$

Fig. 1. Inference rules for ranked conversions and rewrite steps

$\mathcal{V}(r) \subseteq \{y_1, \ldots, y_n\}$ (iii) $\{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_n\} = \emptyset$, and (iv) $x_i \neq x_j$ for all $1 \leq i, j \leq n$ such that $i \neq j$. Here, note that some variables in $y_1, \ldots, y_n$ can be identical. A finite set of LR-separated conditional rewrite rules is called an *LR-separated conditional term rewriting system* (*LR-separated CTRS*, for short). An LR-separated conditional rewrite rule $l \to r \Leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n$ is *non-duplicating* if $|r|_y \leq |y_1, \ldots, y_n|_y$ for all $y \in \mathcal{V}(r)$.

The LR-separated conditional linearization translates TRSs to LR-separated CTRSs: Let $C[y_1, \ldots, y_n] \to r$ be a rewrite rule, where $\mathcal{V}(C) = \emptyset$. Here, some variables in $y_1, \ldots, y_n$ may be identical. Then, we take fresh distinct n variables $x_1, \ldots, x_n$, and construct $C[x_1, \ldots, x_n] \to r \Leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n$ as the result of the translation. It is easily seen that the result is indeed an LR-separated conditional rewrite rule. It is also easily checked that if the rewrite rule is non-duplicating then so is the result of the translation (as an LR-separated conditional rewrite rule). The LR-separated conditional linearization $\mathcal{R}^S$ of a TRS $\mathcal{R}$ is obtained by applying the translation to each rule.

It is shown in [21] that semi-equational non-duplicating LR-separated CTRSs are confluent if their CCPs satisfy some condition, which makes the rewrite steps 'weight-decreasing joinable'. By applying the criterion to LR-separated conditional linearization of TRSs, they obtained a criterion of UNC for non-duplicating TRSs. Note that rewriting in LR-separated CTRSs is (highly) non-deterministic; even reducts of rewrite steps at the same position by the same rule are generally not unique, not only reflecting semi-equational evaluation of the conditional part but also by the $\mathcal{V}(l) \cap \mathcal{V}(r) = \emptyset$ for LR-separated conditional rewrite rule $l \to r \Leftarrow \Gamma$. Thus, how to effectively check the sufficient condition of weight-decreasing joinability is not very clear, albeit it is mentioned in [21] that the decidability is clear.

For obtaining an algorithm for computing the criterion, we introduce ternary relations parameterized by an LR-separated CTRS $\mathcal{R}$ and $n \in \mathbb{N}$ as follows.

Definition 3. *The derivation rules for $\Gamma \Vdash_{\mathcal{R}} u \sim_n v$ and $\Gamma \Vdash_{\mathcal{R}} u \to_n v$ are given in Figure 1. Here, $n \in \mathbb{N}$ and Γ is a multiset of equations.*

Intuitively, $\Gamma \Vdash_{\mathcal{R}} u \sim_n v$ means that $u \overset{*}{\leftrightarrow}_{\mathcal{R}} v$ using the assumption Γ where the number of rewrite steps is n in total (i.e. including those used in checking conditions). Main differences to the relation $\underset{\Gamma}{\sim}$ in [21] are twofold:

1. Instead of considering a special constant $\bullet$, we use an index of natural number. The number of $\bullet$ corresponds to the index number.
2. Auxiliary equations in Γ are allowed in our notation of $\Gamma \Vdash_{\mathcal{R}} u \sim_n v$ (i.e. not all equations in Γ need not be used). On the contrary, Γ in $\underset{\Gamma}{\sim}$ in [21] does not allow auxiliary equations in Γ.

The former is useful to designing the effective procedure to check the UNC criteria presented below. The latter is convenient to prove the satisfiability of constraints on such expressions.

The following slightly generalizes the main result of [21].

Theorem 3. *A semi-equational non-duplicating LR-separated CTRS $\mathcal{R}$ is weight-decreasing joinable if for any CCP $\Gamma \Rightarrow \langle s, t\rangle$ of $\mathcal{R}$, either (i) $\Gamma \Vdash_{\mathcal{R}} s \sim_{\leq 1} t$, (ii) $\Gamma \Vdash_{\mathcal{R}} s \leftrightarrow_2 t$, or (iii) $\Gamma \Vdash_{\mathcal{R}} s \to_i \circ \sim_j t$ with $i + j \leq 2$ and $\Gamma \Vdash_{\mathcal{R}} t \to_{i'} \circ \sim_{j'} s$ with $i' + j' \leq 2$.*

Thus, non-duplicating TRSs $\mathcal{R}$ are UNC if all CCPs of $\mathcal{R}^S$ satisfy some of these (i)–(iii).

Thanks to our new formalization, decidability of the condition easily follows.

Theorem 4. *The condition of Theorem 3 is decidable.*

Proof. We show that each condition (i)–(iii) is decidable. Let Γ be a (finite) multiset of equations, s, t terms, and $\boldsymbol{s}, \boldsymbol{t}$ sequences of terms. The claim follows by showing the following series of sets are finite and effectively constructed one by one: (a) $\mathrm{SIM}_0(\Gamma, s) = \{\langle \Sigma, t\rangle \mid \Gamma \backslash \Sigma \Vdash_{\mathcal{R}} s \sim_0 t\}$, (b) $\mathrm{SIM}_0(\Gamma, \boldsymbol{s}) = \{\langle \Sigma, \boldsymbol{t}\rangle \mid \Gamma \backslash \Sigma \Vdash_{\mathcal{R}} \boldsymbol{s} \sim_0 \boldsymbol{t}\}$, (c) $\mathrm{RED}_1(\Gamma, s, t) = \{\Sigma \mid \Gamma \backslash \Sigma \Vdash_{\mathcal{R}} s \to_1 t\}$, (d) $\mathrm{SRS}_{010}(\Gamma, s, t) = \{\Sigma \mid \Gamma \backslash \Sigma \Vdash_{\mathcal{R}} s \sim_0 \circ \to_1 \circ \sim_0 t\}$, (e) $\mathrm{SIM}_1(\Gamma, s, t) = \{\Sigma \mid \Gamma \backslash \Sigma \Vdash_{\mathcal{R}} s \sim_1 t\}$, (f) $\mathrm{SIM}_1(\Gamma, \boldsymbol{s}, \boldsymbol{t}) = \{\Sigma \mid \Gamma \backslash \Sigma \Vdash_{\mathcal{R}} \boldsymbol{s} \sim_1 \boldsymbol{t}\}$, and (g) $\mathrm{RED}_2(\Gamma, s, t) = \{\Sigma \mid \Gamma \backslash \Sigma \Vdash_{\mathcal{R}} s \to_2 t\}$. ◻

Example 5. Let

$$\mathcal{R} = \left\{ \begin{array}{lll} f(x,x) & \to h(x, f(x,b)) & \quad f(g(y),y) \to h(y, f(g(y), c(b))) \\ h(c(x),b) & \to h(b,b) & \quad c(b) \qquad\; \to b \end{array} \right\}$$

Since $\mathcal{R}$ is non-terminating, non-shallow, and non-right-ground, previous decidability results for UNC does not apply. Furthermore, since $\mathcal{R}$ is overlapping and duplicating, previous sufficient criteria for UNC does not apply. Also, previous modularity results for UNC does not properly decompose $\mathcal{R}$. By conditional linearization, we obtain

$$\mathcal{R}^S = \left\{ \begin{array}{lll} f(x_1, x_2) & \to h(x, f(x,b)) & \Leftarrow x_1 \approx x, x_2 \approx x \\ f(g(y_1), y_2) & \to h(y, f(g(y), c(b))) & \Leftarrow y_1 \approx y, y_2 \approx y \\ h(c(x), b) & \to h(b,b) & \quad c(b) \ \to \ b \end{array} \right\}.$$

We have an overlay CCP $\Gamma \Rightarrow \langle h(x, f(x,b)), h(y, f(g(y), c(b)))\rangle$, where $\Gamma = \{(a) : y_1 \approx y,\ \ (b) : y_2 \approx y,\ \ (c) : g(y_1) \approx x,\ \ (d) : y_2 \approx x\}$. (Another one is its symmetric version.) Let $s = h(y, f(g(y), c(b)))$ and $t = h(x, f(x, b)))$. To check the criteria of Theorem 3, we start computing $\mathrm{SIM}_0(\Gamma, s)$ and $\mathrm{SIM}_0(\Gamma, t)$. For example, the former equals to

$$\left\{\begin{array}{ll}
\langle\{(a),(b),(c),(d)\}, h(y, f(g(y), c(b)))\rangle & \langle\{(b),(c),(d)\}, h(y_1, f(g(y), c(b)))\rangle \\
\langle\{(b),(c),(d)\}, h(y, f(g(y_1), c(b)))\rangle & \langle\{(b),(d)\}, h(y, f(x, c(b)))\rangle \\
\langle\{(a),(c),(d)\}, h(y_2, f(g(y), c(b)))\rangle & \langle\{(a),(c),(d)\}, h(y, f(g(y_2), c(b)))\rangle \\
\langle\{(a),(c)\}, h(x, f(g(y), c(b)))\rangle & \langle\{(a),(c)\}, h(y, f(g(x), c(b)))\rangle \\
\langle\{(c),(d)\}, h(y_1, f(g(y_2), c(b)))\rangle & \langle\{(c),(d)\}, h(y_2, f(g(y_1), c(b)))\rangle \\
\langle\{(c)\}, h(y_1, f(g(x), c(b)))\rangle & \langle\{(c)\}, h(x, f(g(y_1), c(b)))\rangle \\
\langle\{(d)\}, h(y_2, f(x, c(b)))\rangle & \langle\emptyset, h(x, f(x, c(b)))\rangle
\end{array}\right\}.$$

We now can check $s \sim_0 t$ does not hold by $\langle\Gamma', t\rangle \in \mathrm{SIM}_0(\Gamma, s)$ for no Γ'. To check $\Gamma \Vdash s \to_1 t$, we compute $\mathrm{RED}_1(\Gamma, s, t)$. For this, we check there exist a context C and substitution θ and rule $l \to r \Leftarrow \Gamma \in \mathcal{R}^S$ such that $s = C[l\theta]$ and $t = C[r\theta]$. In our case, it is easy to see $\mathrm{RED}_1(\Gamma, s, t) = \emptyset$. Next to check $\Gamma \Vdash s \sim_1 t$, we compute $\mathrm{SRS}_{010}(\Gamma, s, t)$. This is done by, for each $\langle\Gamma', s'\rangle \in \mathrm{SIM}_0(\Gamma, s)$, computing $\langle\Sigma, t'\rangle \in \mathrm{SIM}_0(\Gamma', t)$ and check there exists $\Sigma \in \mathrm{RED}_1(\Sigma', s', t')$. In our case, for $\langle\emptyset, h(x, f(x, c(b)))\rangle \in \mathrm{SIM}_0(\Gamma, s)$ we have $\langle\emptyset, t\rangle \in \mathrm{SIM}_0(\emptyset, t)$, and $\emptyset \in \mathrm{RED}_1(\emptyset, h(x, f(x, c(b))), t)$. Thus, we know $h(x, f(x, c(b))) \to_1 h(x, f(x, b))$. Hence, for these overlay CCPs, we have $y_1 \approx y, y_2 \approx y, g(y_1) \approx x, y_2 \approx x \Vdash_{\mathcal{R}} h(y, f(g(y), c(b))) \sim_1 h(x, f(x, b))$. We also have $\mathrm{CCP}_{in}(\mathcal{R}^S) = \{\ \emptyset \Rightarrow \langle h(b, b), h(b, b)\rangle\}$. For this inner-outer critical pair, it follows that $\Vdash_{\mathcal{R}} h(b, b) \sim_0 h(b, b)$ using $\langle\emptyset, h(b, b)\rangle \in \mathrm{SIM}_0(\emptyset, h(b, b))$. Thus, from Theorem 3, $\mathcal{R}^S$ is weight-decreasing. Hence, it follows $\mathcal{R}$ is UNC. We remark that, in order to derive $\Vdash_{\mathcal{R}} h(b, b) \sim_0 h(b, b)$, we need the reflexivity rule. However, since the corresponding Definition of $\sim$ in the paper [21] lacks the reflexivity rule, the condition of weight-decreasing in [21] (Definition 9) does not hold for $\mathcal{R}^S$. A part of situations where the reflexivity rule is required is covered by the congruence rule, and the reflexivity rule becomes necessary when there exists a trivial critical pair such as above.

5 Equivalent transformation for UNC

In this section, we present a transformational approach for proving and disproving UNC.

5.1 Equivalent transformation and disproof

Firstly, observe that the conditional linearization does not change the input TRSs if they are left-linear. Thus, the technique has no effects on left-linear rewrite rules. But, as one can easily see, it is not at all guaranteed that left-linear TRSs are UNC.

Now, observe that a key idea in the conditional linearization technique is that the CR property of an approximation of a TRS implies the UNC property of the original TRS. The first method presented in this section is based on

Addition

$$\frac{\mathcal{R}}{\mathcal{R} \cup \{l \to r\}}\ l \notin \mathrm{NF}(\mathcal{R}), l \overset{*}{\leftrightarrow}_{\mathcal{R}} r, \mathcal{V}(r) \subseteq \mathcal{V}(l)$$

Elimination

$$\frac{\mathcal{R} \cup \{l \to r\}}{\mathcal{R}}\ l \notin \mathrm{NF}(\mathcal{R}), l \overset{*}{\leftrightarrow}_{\mathcal{R}} r$$

Reversing

$$\frac{\mathcal{R} \cup \{l \to r\}}{\mathcal{R} \cup \{l \to l, r \to l\}}\ r \notin \mathrm{NF}(\mathcal{R} \cup \{l \to r\}), \mathcal{V}(l) \subseteq \mathcal{V}(r)$$

Disproof-1

$$\frac{\mathcal{R}}{\perp}\ l, r \in \mathrm{NF}(\mathcal{R}), l \overset{*}{\leftrightarrow}_{\mathcal{R}} r, l \neq r$$

Disproof-2

$$\frac{\mathcal{R}}{\perp}\ r \in \mathrm{NF}(\mathcal{R}), l \overset{*}{\leftrightarrow}_{\mathcal{R}} r, \mathcal{V}(r) \not\subseteq \mathcal{V}(l)$$

Fig. 2. Inference rules for equivalent transformation and disproof

the observation that one can also use the approximation other than conditional linearization. To fit our usage, we now slightly modify Proposition 1 to obtain the next two lemmas, whose proofs are easy.

Lemma 1. *Suppose (1)* $\to_0 \subseteq \to_1 \subseteq \overset{*}{\leftrightarrow}_0$ *and (2)* $\mathrm{NF}_0 \subseteq \mathrm{NF}_1$*. Then,* $\mathrm{UNC}(\to_0)$ *iff* $\mathrm{UNC}(\to_1)$*.*

Lemma 2. *Suppose (1)* $\overset{=}{\leftrightarrow}_0 = \overset{=}{\leftrightarrow}_1$ *and (2)* $\mathrm{NF}_0 = \mathrm{NF}_1$*. Then,* $\mathrm{UNC}(\to_0)$ *iff* $\mathrm{UNC}(\to_1)$*.*

These lemmas are made into first three transformation rules in Figure 2.

Definition 4. *Let* $\mathcal{R}$ *be a TRS. We write* $\mathcal{R} \rightsquigarrow \alpha$ *if* α *is obtained by one of the inference rules in Figure 2.*

The next lemma immediately follows from Lemmas 1 and 2.

Lemma 3. *Let* $\mathcal{R}$ *be a TRS and* $l \to r$ *a rewrite rule.*

1. *Suppose* $l \overset{*}{\leftrightarrow}_{\mathcal{R}} r$ *and* $l \notin \mathrm{NF}(\mathcal{R})$*. Then,* $\mathrm{UNC}(\mathcal{R})$ *iff* $\mathrm{UNC}(\mathcal{R} \cup \{l \to r\})$*.*
2. *Suppose* $r \to l$ *is a rewrite rule and* $r \notin \mathrm{NF}(\mathcal{R} \cup \{l \to r\})$*. Then* $\mathrm{UNC}(\mathcal{R} \cup \{l \to r\})$ *iff* $\mathrm{UNC}(\mathcal{R} \cup \{l \to l, r \to l\})$*.*

Applying Lemma 3 (1) to the Addition and Elimination rules, and Lemma 3 (2) to the Reversing rules, we obtain:

Theorem 5. *Let* $\mathcal{R}$ *be a TRS and suppose* $\mathcal{R} \overset{*}{\rightsquigarrow} \mathcal{R}' \neq \perp$*. Then,* $\mathcal{R}'$ *is a TRS, and* $\mathrm{UNC}(\mathcal{R}')$ *iff* $\mathrm{UNC}(\mathcal{R})$*.*

Note that the relation $\rightsquigarrow$ is not well-founded; we will present some strategies for automation in the next subsection. We next show the correctness of the Disproof-1/2 rules.

Theorem 6. *Let* $\mathcal{R}$ *be a TRS and suppose* $\mathcal{R} \overset{*}{\rightsquigarrow} \perp$*. Then* $\neg\mathrm{UNC}(\mathcal{R})$*.*

Proof. Then we have $\mathcal{R} \stackrel{*}{\rightsquigarrow} \mathcal{R}' \rightsquigarrow \bot$ for some $\mathcal{R}'$. From Theorem 5, we have $\mathrm{UNC}(\mathcal{R}')$ iff $\mathrm{UNC}(\mathcal{R})$. Thus, it remains to show $\neg\mathrm{UNC}(\mathcal{R}')$. Suppose $\mathcal{R}' \rightsquigarrow \bot$ by Disproof-1. Then $l \stackrel{*}{\leftrightarrow}_{\mathcal{R}'} r$, $l, r \in \mathrm{NF}(\mathcal{R}')$, and $l \neq r$. By the definition of UNC, $\mathcal{R}'$ is not UNC. Suppose $\mathcal{R}' \rightsquigarrow \bot$ by Disproof-2. Then $s \stackrel{*}{\leftrightarrow}_{\mathcal{R}'} t \in \mathrm{NF}(\mathcal{R}')$ and $x \in \mathcal{V}(t) \setminus \mathcal{V}(s)$. Take a fresh variable y and let $t' = t\{x := y\}$. Clearly, from $t \in \mathrm{NF}(\mathcal{R}')$ we have $t' \in \mathrm{NF}(\mathcal{R}')$. By $t' \stackrel{*}{\leftrightarrow}_{\mathcal{R}'} s \stackrel{*}{\leftrightarrow}_{\mathcal{R}'} t$, $\mathcal{R}'$ is not UNC. $\square$

5.2 Automation

The correctness of equivalent transformation itself does not give us any hint how to apply such transformations. Below, we give two procedures based on the equivalent transformation.

First one employs the Reversing rule, the Elimination rule, and an ordering $>$ as a heuristic (not to loop).

Definition 5 (Rule reversing transformation). *Let $\mathcal{R}$ be a TRS. We write $\mathcal{R} \hookrightarrow \mathcal{R}'$ if $\mathcal{R}' = (\mathcal{R} \setminus \{l \to r\}) \cup \{l \to l, r \to l\}$ for some $l \to r \in \mathcal{R}$ such that $l < r$, $r \notin \mathrm{NF}(\mathcal{R})$ and $r \to l$ is a rewrite rule, or $\mathcal{R}' = \mathcal{R} \setminus \{l \to r\}$ for some $l \to r \in \mathcal{R}$ such that $l = r$ and $l \notin \mathrm{NF}(\mathcal{R} \setminus \{l \to r\})$. Any transformation $\mathcal{R} \stackrel{*}{\hookrightarrow} \mathcal{R}'$ is called a* rule reversing transformation*.*

It is easy to see that the relation $\hookrightarrow$ is well-founded, by comparing the number of increasing rules (i.e. $l \to r$ such that $l < r$) and the number of rules lexicographically. The correctness follows from Theorem 5.

Theorem 7. *Let $\mathcal{R}'$ be a TRS obtained by a rule reversing transformation from $\mathcal{R}$. Then,* $\mathrm{UNC}(\mathcal{R})$ *iff* $\mathrm{UNC}(\mathcal{R}')$*.*

Next, we consider constructing an approximation $\mathcal{S}$ of a TRS $\mathcal{R}$ by adding auxiliary rules generated by critical pairs. To guide the procedure, we consider two predicates φ and Φ such that the following confluence criterion holds:

> Suppose that TRS $\mathcal{S}$ satisfies $\varphi(\mathcal{S})$. If $\Phi(u, v)$ holds for all critical pairs $\langle u, v\rangle$ of $\mathcal{S}$, then $\mathcal{S}$ has the CR property. (A)

Multiple criteria in this form are known: one can take $\varphi(\mathcal{S})$ and $\Phi(u, v)$ as '$\mathcal{S}$ is left-linear' and '$\langle u, v\rangle$ is development-closed', respectively [16] and as '$\mathcal{S}$ is linear' and '$\langle u, v\rangle$ is strongly closed', respectively [8]. The idea is that if one encounters a critical pair $\langle u, v\rangle$ for which $\Phi(u, v)$ does not hold, then (check whether one can apply Disproof rules and) apply the equivalent transformation so that $\Phi(u, v)$ is satisfied.

Definition 6 (UNC completion procedure).

Input*: TRS $\mathcal{R}$, predicates φ, Φ satisfying* (A)*.*
Output*: UNC or NotUNC or Failure (or may diverge)*

Step 1. *Compute the set* $\mathrm{CP}(\mathcal{R})$ *of critical pairs of $\mathcal{R}$.*

Step 2. *If $\Phi(u,v)$ for all $\langle u,v\rangle \in \mathrm{CP}(\mathcal{R})$ and $\varphi(\mathcal{R})$ then return UNC.*
Step 3. *Let $\mathcal{S} := \emptyset$. For each $\langle u,v\rangle \in \mathrm{CP}(\mathcal{R})$ with $u \neq v$ for which $\Phi(u,v)$ does not hold, do:*
(a) If $u,v \in \mathrm{NF}(\mathcal{R})$, then exit with NotUNC.
(b) If $u \notin \mathrm{NF}(\mathcal{R})$ and $v \in \mathrm{NF}(\mathcal{R})$, then if $\mathcal{V}(v) \not\subseteq \mathcal{V}(u)$ then exit with NotUNC, otherwise update $\mathcal{S} := \mathcal{S} \cup \{u \to v\}$.
(c) If $v \notin \mathrm{NF}(\mathcal{R})$ and $u \in \mathrm{NF}(\mathcal{R})$, then if $\mathcal{V}(u) \not\subseteq \mathcal{V}(v)$ then exit with NotUNC, otherwise update $\mathcal{S} := \mathcal{S} \cup \{v \to u\}$.
(d) If $u,v \notin \mathrm{NF}(\mathcal{R})$ then find w such that $u \xrightarrow{*}_{\mathcal{R}} w$ ($v \xrightarrow{*}_{\mathcal{R}} w$), and $\mathcal{V}(w) \subseteq \mathcal{V}(v)$ (resp. $\mathcal{V}(w) \subseteq \mathcal{V}(v)$). If it succeeds then update $\mathcal{S} := \mathcal{S} \cup \{v \to w\}$.
Step 4. *If $\mathcal{S} = \emptyset$ then return Failure; otherwise update $\mathcal{R} := \mathcal{R} \cup \mathcal{S}$ and go back to Step 1.*

Again, the correctness of the UNC completion procedure follows immediately from Theorems 5 and 6.

Theorem 8. *The UNC completion procedure is correct, i.e. if the procedure returns UNC then* $\mathrm{UNC}(\mathcal{R})$*, and if the procedure returns NotUNC then* $\neg\mathrm{UNC}(\mathcal{R})$.

Example 6. Let $\mathcal{R} = \{a \to a, f(f(x,b),y) \to f(y,b), f(b,y) \to f(y,b), f(x,a) \to b\}$. Since $\mathcal{R}$ is non-terminating, non-shallow, and non-right-ground, previous decidability results for UNC does not apply. Furthermore, since $\mathcal{R}$ is overlapping and $\mathcal{R}^L = \mathcal{R}$ is non-confluent, previous sufficient criteria for UNC does not apply. Also, previous modularity results for UNC does not properly decompose $\mathcal{R}$. Now, let us apply the UNC completion procedure to $\mathcal{R}$ using linear strongly closed criteria for confluence. For this, take $\varphi(\mathcal{R})$ as $\mathcal{R}$ is linear, and $\Phi(u,v)$ as $(u \xrightarrow{*} \circ \xleftarrow{=} v) \wedge (u \xrightarrow{=} \circ \xleftarrow{*} v)$. In Step 3, we find an overlay critical pair $\langle f(a,b), b\rangle$, for which Φ is not satisfied. Since $f(a,b)$ is not normal and b is normal, we go to Step 3(b). Thus, we update $\mathcal{R} := \mathcal{R} \cup \{f(a,b) \to b\}$. Now, the updated $\mathcal{R}$ is linear and strongly closed (and thus, $\mathcal{R}$ is confluent). Hence, the procedure returns UNC at Step 2.

6 Implementation and experiment

We have tested various methods presented so far. The methods used in our experiment are summarized as follows.

(ω) $\mathrm{UNC}(\mathcal{R})$ if $\mathcal{R}$ is non-ω-overlapping.
(pcl) $\mathrm{UNC}(\mathcal{R})$ if $\mathcal{R}^L$ is parallel-closed.
(scl) $\mathrm{UNC}(\mathcal{R})$ if $\mathcal{R}$ is right-linear and $\mathcal{R}^L$ is strongly closed.
(wd) $\mathrm{UNC}(\mathcal{R})$ if $\mathcal{R}^S$ is non-duplicating and weight-decreasing joinable.
(sc) UNC completion using strongly closed critical pairs criterion.
(dc) UNC completion using development-closed critical pairs criterion.
(rr) $\mathrm{UNC}(\mathcal{R})$ if $\mathcal{R}$ is right-reducible.
(cp) $\neg\mathrm{UNC}(\mathcal{R})$ by adhoc search of a counterexample for $\mathrm{UNC}(\mathcal{R})$.

(rev) Rule reversing transformation, combined with other criteria above.

For the implementation of non-ω-overlapping condition, we used unification algorithm over infinite terms in [9]. For **(sc)** and **(dc)**, we approximate $\xrightarrow{*}$ by the development step $\multimap\!\!\!\rightarrow$ (e.g. [16]) in Step 3(d). We employed as the heuristic ordering $>$ for **(rev)** the comparison in terms of size. For **(cp)**, we use an adhoc search based on rule reversing, critical pairs computation, and rewriting.

We tested on the 242 TRSs from the Cops (Confluence Problems) database[5] of which no confluence tool has proven confluence nor terminating at the time of experiment[6]. The motivation of using such testbed is as follows: If a confluent tool can prove CR, then UNC is obtained by confluent tools. If $\mathcal{R}$ is terminating then CR($\mathcal{R}$) iff UNC($\mathcal{R}$), and thus the result follows also from the result of confluence tools. Thus, we here evaluate our UNC techniques on such testbed.

without ***(rev)***	(ω)	**(pcl)**	**(scl)**	**(wd)**	**(sc)** 1/2/3	**(dc)** 1/2/3	**(rr)**	**(cp)**	*all*
`YES`	10	8	3	3	4/10/12	3/9/12	45	0	62
`NO`	0	0	0	0	24/49/59	24/49/59	0	68	87
`YES`+`NO`	10	8	0	0	28/59/71	27/58/71	45	68	149
timeout (60s)	0	0	0	0	13/20/53	15/23/70	0	0	–
time (min)	0	0	0	0	13/21/60	16/25/79	0	2	–
with ***(rev)***	(ω)	**(pcl)**	**(scl)**	**(wd)**	**(sc)** 1/2/3	**(dc)** 1/2/3	**(rr)**	**(cp)**	*all*
`YES`	6	4	1	1	26/44/47	26/37/41	45	0	75
`NO`	0	0	0	0	25/52/60	25/53/61	0	60	84
`YES`+`NO`	6	4	1	1	51/96/107	51/90/102	45	60	159
timeout (60s)	0	0	0	3	14/19/47	14/19/60	0	0	–
time (min)	0	0	0	4	15/20/54	15/21/70	0	0	–
both	(ω)	**(pcl)**	**(scl)**	**(wd)**	**(sc)** 1/2/3	**(dc)** 1/2/3	**(rr)**	**(cp)**	*all*
`YES`+`NO`	10	8	3	3	53/102/112	52/96/106	45	68	171

Table 1. Test on presented criteria

In Table 1, we summarize the results. Our test is performed on a PC with 2.60GHz cpu with 4G of memory. The column headings show the technique used. The number of examples for which UNC is proved (disproved) successfully is shown in the row titled '`YES`' (resp. '`NO`'). In the columns below **(sc)** and **(dc)**, we put $l/n/m$ where each l, n, m denotes the scores for the 1-round (2-rounds, 3-rounds) UNC completion. The columns below '*all*' show the numbers of examples succeeded in any of the methods.

The columns below the row headed '*with* ***(rev)***' are the results for which methods are applied after the rule reversing transformation. The columns below

[5] Cops can be accessed from http://cops.uibk.ac.at/, which consists of 1137 problems at the time of experiment.

[6] This was obtained by a query '`trs !confluent !terminating`' in Cops at the time of experiment.

the row headed '*both*' show the numbers of examples succeeded by each technique, where the techniques are applied to both of the original TRSs and the TRSs obtained by the rule reversing transformation.

3 rounds UNC completions **(sc)**, **(dc)** with rule reversing are most effective, but they are also the most time consuming. Simple methods **(rr)**, **(cp)** are also effective for not few examples. Although there is only a small number of examples for which criteria based on conditional linearization are effective, but their checks are fast compared to the UNC completions. Rule reversing **(rev)** is only worth incorporated for UNC completions. For other methods, the rule reversing make the methods less effective; for methods (ω), **(pcl)**, **(scl)** and **(wd)**, this is because the rule reversing transformation generally increases the number of the rules. In total, the UNC property of the 171 problems out of 242 problems have been solved by presented methods. The details of the experiment are found in http://www.nue.ie.niigata-u.ac.jp/tools/acp/experiments/frocos19/.

7 Tool

ACP originally intends to (dis)prove confluence of TRSs [3]. ACP integrates multiple direct criteria for guaranteeing confluence of TRSs; it also incorporates several divide-and-conquer criteria. We have extended it to also deal with (dis)proving the UNC property of TRSs.

Like its confluence proving counterpart, ACP first tries to decompose the UNC problem of the given TRS into those of smaller components. For this, one can use the following modularity results on the UNC property, where we refer to [3] for the terminology:

Proposition 5 ([2]). *Suppose* $\{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$ *is a persistent decomposition of* $\mathcal{R}$. *Then,* $\bigcup_i \mathcal{R}_i$ *is UNC if and only if so is each* $\mathcal{R}_i$.

Proposition 6 ([1]). *Suppose* $\{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$ *is a layer-preserving decomposition of* $\mathcal{R}$. *Then,* $\bigcup_i \mathcal{R}_i$ *is UNC if and only if so is each* $\mathcal{R}_i$.

After possible decomposition, multiple direct criteria are tried for each component. For the direct criteria, we have incorporated (ω), **(pcl)**, **(scl)**, **(wd)**, **(rr)**, **(cp)** without rule reversing, and **(sc)**3 and **(dc)**3 with rule reversing. These methods are tried one method after another. We also add yet another UNC check, namely that after the Steps 1–3 of the UNC completion using development-closed critical pairs criterion, the confluence check in ACP is performed.

Other tools that support UNC (dis)proving include CSI [14] which is a confluence prover supporting UNC proof for non-ω-overlapping TRSs and a decision procedure of UNC for ground TRSs (at the time of CoCo 2018[7]), and FORT [18] which implements decision procedure for first-order theory of left-linear right-ground TRSs based on tree automata. Our new methods are also effective for TRSs outside the class of non-ω-overlapping TRSs and that of left-linear right-ground TRSs. We use the same testbed in the previous section, to compare our

[7] The recent version of CSI had been extended with some other techniques.

tool with the latest versions of CSI (ver. 1.2.2) and FORT (ver. 2.1), also test the effect of the divide-and-conquer criteria. The result is shown in the Table 3.

	ACP	ACP$_{(\text{direct})}$	CSI	FORT
YES	83	83	86	38
NO	92	92	65	34
time	62m	62m	78m	2m

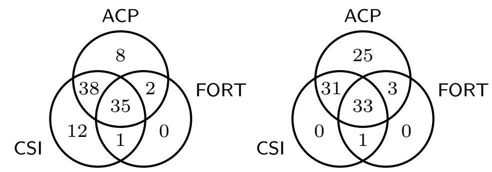

Table 2. Comparison of UNC tools

There is no example in the testbed that fails when decomposition techniques are inactivated (ACP (direct)). For the next example, however, our tool succeeds only if the decomposition techniques are activated.

Example 7. Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, where $\mathcal{R}_1 = \{f(f(x,y),z) \to f(f(x,z),f(y,z))\}$ and $\mathcal{R}_2 = \{\ @(@(@(S,x),y),z) \ \to\ @(@(x,z),@(y,z)),\ @(@(K,x),y) \ \to\ x,$ $@(I,x) \to x,\ @(@(D,x),x) \to x\ \}$. By the persistency decomposition, UNC($\mathcal{R}$) follows UNC($\mathcal{R}_1$) and UNC($\mathcal{R}_2$). Since $\mathcal{R}_1$ is right-reducible, UNC($\mathcal{R}_1$) holds. Since $\mathcal{R}_2$ is non-ω-overlapping, UNC($\mathcal{R}_2$) holds. Thus, one obtains UNC($\mathcal{R}$).

The techniques in the present paper mainly contributed to make our tool ACP win the UNC category of CoCo 2018. The details of the competition can be seen at http://project-coco.uibk.ac.at/2018/. The version of ACP for CoCo 2018 (ver. 0.62) is downloadable from http://www.nue.ie.niigata-u.ac.jp/tools/acp/.

8 Conclusion

In this paper, we have studied automated methods for (dis)proving the UNC property of TRSs. We have presented some new methods for (dis)proving the UNC property of TRSs. Presented methods have been implemented in our tool ACP based on divide-and-conquer criteria.

Acknowledgements

Thanks are due to the anonymous reviewers of the previous versions of the paper. This work is partially supported by JSPS KAKENHI No. 18K11158.

References

1. Aoto, T., Toyama, Y.: Top-down labelling and modularity of term rewriting systems. Research Report IS-RR-96-0023F, School of Information Science, JAIST (1996)

2. Aoto, T., Toyama, Y.: On composable properties of term rewriting systems. In: Proc. of 6th ALP and 3rd HOA. LNCS, vol. 1298, pp. 114–128. Springer-Verlag (1997)
3. Aoto, T., Yoshida, Y., Toyama, Y.: Proving confluence of term rewriting systems automatically. In: Proc. of 20th RTA. LNCS, vol. 5595, pp. 93–102. Springer-Verlag (2009)
4. Baader, F., Nipkow, T.: Term Rewriting and All That. Cambridge University Press (1998)
5. Bergstra, J.A., Klop, J.W.: Conditional rewrite rules: confluence and termination. Journal of Computer and System Sciences **32**, 323–362 (1986)
6. Dauchet, M., Heuillard, T., Lescanne, P., Tison, S.: Decidability of the confluence of finite ground term rewrite systems and of other related term rewrite systems. Information and Computation **88**, 187–201 (1990)
7. Gramlich, B.: Modularity in term rewriting revisited. Theoretical Computer Science **464**, 3–19 (2012)
8. Huet, G.: Confluent reductions: abstract properties and applications to term rewriting systems. Journal of the ACM **27**(4), 797–821 (1980)
9. Jaffar, J.: Efficient unification over infinite terms. New Generation Computing **2**, 207–219 (1984)
10. Kahrs, S., Smith, C.: Non-ω-overlapping TRSs are UN. In: Proc. of 1st FSCD. LIPIcs, vol. 52, pp. 22:1–17. Schloss Dagstuhl (2016)
11. Klein, D., Hirokawa, N.: Confluence of non-left-linear TRSs via relative termination. In: Proc. of 18th LPAR. LNCS, vol. 7180, pp. 258–273. Springer-Verlag (2012)
12. Klop, J.: Combinatory Reduction Systems, Mathematical Centre Tracts, vol. 127. CWI, Amsterdam, Holland (1980)
13. Klop, J., de Vrijer, R.: Extended term rewrite systems. In: Proc. of 2nd CTRS. LNCS, vol. 516, pp. 26–50. Springer-Verlag (1990)
14. Nagele, J., Felgenhauer, B., Middeldorp, A.: CSI: new evidence - a progress report. In: Proc. of 26th CADE. LNAI, vol. 10395, pp. 385–397. Springer-Verlag (2017)
15. O'Donnell, M.: Computing in systems described by equations, LNCS, vol. 58. Springer-Verlag (1977)
16. van Oostrom, V.: Developing developments. Theoretical Computer Science **175**(1), 159–181 (1997)
17. Radcliffe, N.R., Moreas, L.F.T., Verma, R.M.: Uniqueness of normal forms for shallow term rewrite systems. ACM Transactions on Computational Logic **18**(2), 17:1–17:20 (2017)
18. Rapp, F., Middeldorp, A.: Automating the first-order theory of rewriting for left-linear right-ground rewrite systems. In: Proc. of 1st FSCD. LIPIcs, vol. 52, pp. 36:1–17. Schloss Dagstuhl (2016)
19. Toyama, Y.: Commutativity of term rewriting systems. In: Fuchi, K., Kott, L. (eds.) Programming of Future Generation Computers II, pp. 393–407. North-Holland (1988)
20. Toyama, Y.: Confluent term rewriting systems (invited talk). In: Proc. of 16th RTA. LNCS, vol. 3467, p. 1. Springer-Verlag (2005), slides at http://www.nue.ie.niigata-u.ac.jp/toyama/user/toyama/slides/toyama-RTA05.pdf
21. Toyama, Y., Oyamaguchi, M.: Conditional linearization of non-duplicating term rewriting systems. IEICE Transactions on Information and Systems **E84-D**(4), 439–447 (2001)
22. de Vrijer, R.: Conditional linearization. Indagationes Mathematicae **10**(1), 145–159 (1999)

A Omitted Proofs

Proof (of Proposition 2). Suppose $s \stackrel{*}{\leftrightarrow}_{\mathcal{R}} t$, $s, t \in \mathrm{NF}(\mathcal{R})$ and $s \neq t$. Then from $s \neq t$, we have $s \stackrel{+}{\leftrightarrow}_{\mathcal{R}} t$, and thus $s \leftrightarrow_{\mathcal{R}} s' \stackrel{*}{\leftrightarrow}_{\mathcal{R}} t$ for some s'. If $s \to_{\mathcal{R}} s'$ then this contradicts $s \in \mathrm{NF}(\mathcal{R})$. If $s' \to_{\mathcal{R}} s$ then $s' = C[l\theta]$ and $s = C[r\theta]$ for some $l \to r \in \mathcal{R}$, and hence from $r \notin \mathrm{NF}(\mathcal{R})$ we know $s \notin \mathrm{NF}(\mathcal{R})$. This is again a contradiction. $\square$

In the proofs below, we use the notion of *subterm occurrences*. Subterm occurrences α and β are identified if and only if α and β occur at the same position. Thus, $\alpha = \beta$ also implies they are identical as a term, but distinct occurrences of the same subterm are regarded as not identical. We write $\alpha \subseteq \beta$ if the subterm occurrence α is contained in the subterm occurrence β; we write $\alpha \subset \beta$ if $\alpha \neq \beta$ in addition. It is essential in the proof of Theorem 1 to use the subterm occurrences but not the subterms in the definition of Red_{in}.

We first prepare two lemmas to present a proof of Theorem 1.

Lemma 4. *Let $\mathcal{R}$ be a semi-equational CTRS and $l \to r \Leftarrow \Gamma \in \mathcal{R}$ be left-linear. Suppose $s \ {}_{P}\!\!\leftarrow\!\!\!\!\|\!\!- \ l\theta \to_{\epsilon, l\to r\Leftarrow\Gamma} r\theta$, and any redex occurrence of $l\theta \ -\!\!\|\!\!\!\!\rightarrow_{P} s$ is contained in a subterm occurrence of $\theta(x)$ in $l\theta$ for some $x \in \mathcal{V}(l)$. Then there exists t such that $s \to_{\epsilon, l\to r\Leftarrow\Gamma} t \leftarrow\!\!\!\!\|\!\!- r\theta$. Furthermore, if r is linear and $|P| = 1$ then $s \to_{\epsilon, l\to r\Leftarrow\Gamma} t \stackrel{=}{\leftarrow} r\theta$.*

Proof. Let $P = \{p_1, \ldots, p_k\}$, and, for each $1 \leq i \leq k$, let α_i be the subterm occurrence in $l\theta$ at p_i and β_i be the subterm occurrence in s at p_i. For each $x \in \mathcal{V}(l)$, let $\theta(x) = C_x[\alpha_{i_1}, \ldots, \alpha_{i_m}]$ with all $\alpha_1, \ldots, \alpha_k$ in $\theta(x)$ displayed. Take a substitution θ' such as $\theta'(x) = C_x[\beta_{i_1}, \ldots, \beta_{i_m}]$. Then, we have $s = l\theta'$ by linearity of l, and moreover, $\theta'(y) \stackrel{*}{\leftrightarrow} \theta(y)$ for all $y \in \mathcal{V}$ by definition. From the latter and $\vdash_{\mathcal{R}} \Gamma\theta$, we obtain $\vdash_{\mathcal{R}} \Gamma\theta'$. Thus, $s = l\theta' \to_{\epsilon, l\to r\Leftarrow\Gamma} r\theta'$. Let $r = C'[x_1, \ldots, x_n]$ with all variable occurrences in r displayed. Then $r\theta = C'[x_1\theta, \ldots, x_n\theta] \ -\!\!\|\!\!\!\!\rightarrow C'[x_1\theta', \ldots, x_n\theta'] = r\theta'$. Thus, $s \to_{\epsilon, l\to r\Leftarrow\Gamma} r\theta' \leftarrow\!\!\!\!\|\!\!- r\theta$. Suppose in addition that r is linear and $k = 1$. Then, $\theta(x) = C_x[\alpha_1]$ for some $x \in \mathcal{V}(l)$ and $\theta(y) = C_y$ for all $y \in \mathcal{V}(l)$ such that $y \neq x$. Thus, $r\theta = C'[x_1\theta, \ldots, x_n\theta] \stackrel{=}{\to} C'[x_1\theta', \ldots, x_n\theta'] = r\theta'$ by the linearity of r. $\square$

Lemma 5. *Let $\mathcal{R}$ be a semi-equational CTRS of type 1 and $l_1 \to r_1 \Leftarrow \Gamma_1 \in \mathcal{R}$. Suppose $s \ {}_{p}\!\!\leftarrow l_1\theta \to_{\epsilon, l_1\to r_1\Leftarrow\Gamma_1} r_1\theta$, and the redex occurrence of $l_1\theta \to_p s$ is not contained in any subterm occurrence of $\theta(x)$ $(x \in \mathcal{V}(l_1))$ in $l_1\theta$. Then $s = r_1\theta$ or $s \leftarrow l_1\theta \to r_1\theta$ is an instance of a CCP $\Sigma \Rightarrow \langle u, v\rangle$ i.e. there exists some substitution σ such that $s = u\sigma$, $r_1\theta = v\sigma$ and $\vdash_{\mathcal{R}} \Sigma\sigma$.*

Proof. Let $s \ {}_{p, l_2\to r_2\Leftarrow\Gamma_2}\!\!\leftarrow l_1\theta$. W.l.o.g. assume $\mathcal{V}(l_1 \to r_1 \Leftarrow \Gamma_1) \cap \mathcal{V}(l_2 \to r_2 \Leftarrow \Gamma_2) = \emptyset$. Then we can let $l_1\theta = l_1[l_2]_p\theta$ and $s = l_1\theta[r_2\theta]_p$, and assume $\vdash_{\mathcal{R}} \Gamma_1\theta, \Gamma_2\theta$. By the condition $p \in \mathrm{Pos}_{\mathcal{F}}(l_1)$, we have $l_1\theta|_p = l_1|_p\theta = l_2\theta$, and thus $l_1|_p$ and l_2 are unifiable. Then, there exists an mgu ρ of $l_1|_p$ and l_2 and a substitution σ such that $\sigma \circ \rho = \theta$. If $l_1 \to r_1 \Leftarrow \Gamma_1$ and $l_2 \to r_2 \Leftarrow \Gamma_2$ are identical and $p = \epsilon$, then by our assumption that $\mathcal{R}$ has type 1, we have $s = r_1\theta$.

Otherwise, there exists a CCP $\Gamma_1\rho, \Gamma_2\rho \Rightarrow \langle l_1[r_2]_p\rho, r_1\rho\rangle$ of $\mathcal{R}$. Then we have $s = l_1[r_2]_p\theta = l_1[r_2]_p\rho\sigma = u\sigma$, $r_1\theta = r_1\rho\sigma = v\sigma$, and $\Gamma_1\theta \cup \Gamma_2\theta = (\Gamma_1\rho \cup \Gamma_2\rho)\sigma = \Sigma\sigma$. Thus, from $\vdash_{\mathcal{R}} \Gamma_1\theta, \Gamma_2\theta$, it follows $\vdash_{\mathcal{R}} \Sigma\sigma$. $\square$

Proof (of Theorem 1). We show the claim $t \mathrel{-\!\!\|\!\!\!\rightarrow} t_1$ and $t \mathrel{-\!\!\|\!\!\!\rightarrow} t_2$ imply $t_1 \mathrel{-\!\!\|\!\!\!\rightarrow}^* t_3$ and $t_2 \mathrel{-\!\!\|\!\!\!\rightarrow} t_3$ for some t_3. In fact, the proof is almost same as that of the criteria for TRSs. The only essential difference is captured by Lemmas 4 and 5. For such parallel peak, let $t \mathrel{-\!\!\|\!\!\!\rightarrow}_{P_1} t_1$ with $P_1 = \{p_{11}, \dots, p_{1m}\}$ and $t \mathrel{-\!\!\|\!\!\!\rightarrow}_{P_2} t_2$ with $P_2 = \{p_{21}, \dots, p_{2n}\}$. We set subterm occurrences $\alpha_i = t|_{p_{1i}}$ for $i = 1, \dots, m$ and $\beta_j = t|_{p_{2j}}$ for $j = 1, \dots, n$. Let $t = C_1[\alpha_1, \dots, \alpha_m]_{p_{11},\dots,p_{1m}} = C_2[\beta_1, \dots, \beta_n]_{p_{21},\dots,p_{2n}}$. Let $t|_{p_k} = l_k\sigma_k$ with $l_k \to r_k \Leftarrow \Gamma_k \in \mathcal{R}$. Then, we have $t_1 = C_1[r_{11}\sigma_{11}, \dots, r_{1m}\sigma_{1m}]$ $t_2 = C_2[r_{21}\sigma_{21}, \dots, r_{2n}\sigma_{2n}]$, and $\vdash_{\mathcal{R}} \Gamma_k\sigma_k$ for all $p_k \in P_1 \cup P_2$. Let

$$\begin{aligned}
\mathit{Red}_{in}(t_1 \mathrel{\leftarrow\!\!\!\|\!\!-} t \mathrel{-\!\!\|\!\!\!\rightarrow} t_2) &= \{\alpha_i \mid \exists j.\alpha_i \subset \beta_j\} \uplus \{\beta_j \mid \exists i.\beta_j \subseteq \alpha_i\}\\
\mathit{Red}_{out}(t_1 \mathrel{\leftarrow\!\!\!\|\!\!-} t \mathrel{-\!\!\|\!\!\!\rightarrow} t_2) &= \{\alpha_i \mid \forall j.\alpha_i \not\subset \beta_j\} \uplus \{\beta_j \mid \forall i.\beta_j \not\subseteq \alpha_i\}
\end{aligned}$$

Let us denote by $|t|$ the size of a term t, and $|M| = \sum_{t\in M} |t|$ for a term multiset M. Let $I = \mathit{Red}_{in}(t_1 \mathrel{\leftarrow\!\!\!\|\!\!-} t \mathrel{-\!\!\|\!\!\!\rightarrow} t_2)$. The proof of the claim is by induction on $|I|$.

- Case $|I| = 0$. Then for all $p_{k_1}, p_{k_2} \in P_1 \cup P_2$, $k_1 \neq k_2$ implies $p_{k_1} \parallel p_{k_2}$. For notational simplicity, we only consider the case $t = C[\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n]$, $t_1 = C[\alpha'_1, \dots, \alpha'_m, \beta_1, \dots, \beta_n]$, $t_2 = C[\alpha_1, \dots, \alpha_m, \beta'_1, \dots, \beta'_n]$, with $\alpha_i \to \alpha'_i$ $(1 \le i \le m)$ and $\beta_j \to \beta'_j$ $(1 \le j \le n)$. Let $t_3 = C[\alpha'_1, \dots, \alpha'_m, \beta'_1, \dots, \beta'_n]$, Then $t_1 \mathrel{-\!\!\|\!\!\!\rightarrow}_{P_2} t_3$ and $t_2 \mathrel{-\!\!\|\!\!\!\rightarrow}_{P_1} t_3$.
- Case $|I| > 0$.
  Let $\gamma_1, \dots, \gamma_h$ be subterm occurrences of the term t contained in $\mathit{Red}_{out}(t_1 \mathrel{\leftarrow\!\!\!\|\!\!-} t \mathrel{-\!\!\|\!\!\!\rightarrow} t_2)$.
  Then we can write $t = C'[\gamma_1, \dots, \gamma_h]$, $t_1 = C'[\gamma_{11}, \dots, \gamma_{1h}]$, $t_2 = C'[\gamma_{21}, \dots, \gamma_{2h}]$, where, for each $1 \le k \le h$, $\gamma_k \mathrel{-\!\!\|\!\!\!\rightarrow} \gamma_{1k}$ and $\gamma_k \mathrel{-\!\!\|\!\!\!\rightarrow} \gamma_{2k}$ with one of them being a root step. It is sufficient to show there are $\gamma'_1, \dots, \gamma'_h$ such that $\gamma_{1k} \mathrel{-\!\!\|\!\!\!\rightarrow}^* \gamma'_k$ and $\gamma_{2k} \mathrel{-\!\!\|\!\!\!\rightarrow} \gamma'_k$ for each $1 \le k \le h$.
  Suppose $1 \le k \le h$.
  - Let us consider the case $\gamma_k \mathrel{-\!\!\|\!\!\!\rightarrow}_{\{\epsilon\}} \gamma_{1k}$ and $\gamma_k \mathrel{-\!\!\|\!\!\!\rightarrow}_P \gamma_{2k}$. Then there exist $l \to r \Leftarrow \Gamma \in \mathcal{R}$ and θ such that $\gamma_k = l\theta$ and $\gamma_{1k} = r\theta$ and $\vdash_{\mathcal{R}} \Gamma\theta$. Let $\gamma_k = \hat{C}[\hat{\gamma}_1, \dots, \hat{\gamma}_g]$ where the subterm occurrences $\hat{\gamma}_1, \dots, \hat{\gamma}_g$ are at the respective positions in P. Then we can let $\gamma_{2k} = \hat{C}[\hat{\gamma}'_1, \dots, \hat{\gamma}'_g]$ with $\hat{\gamma}_i \to \hat{\gamma}'_i$ for each $1 \le i \le g$.
    First, consider the case that that for each $\hat{\gamma}_i$, there exists $x \in \mathcal{V}(l)$ such that $\hat{\gamma}_i$ is contained in some $\theta(x)$. Then, by Lemma 4, $\gamma_{2k} \to \circ \mathrel{\leftarrow\!\!\!\|\!\!-} \gamma_{1k}$. Otherwise, there exists some $1 \le i \le g$ such that $\hat{\gamma}_i$ is contained in $\theta(x)$ for no $x \in \mathcal{V}(l)$. Let p be the position of $\hat{\gamma}_i$ in γ_k. Then we have $\gamma_k \to_p \gamma_k[\hat{\gamma}'_i]_p \mathrel{-\!\!\|\!\!\!\rightarrow}_{P\setminus\{p\}} \gamma_{2k}$. Then, by Lemma 5, $\langle \gamma_k[\hat{\gamma}'_i], \gamma_{1k}\rangle$ is an instance of some CCP $\Gamma \Rightarrow \langle u, v\rangle$, i.e. there exists some σ such that $\gamma_k[\hat{\gamma}'_i] = u\sigma$, $\gamma_{1k} = v\sigma$ and $\vdash_{\mathcal{R}} \Gamma\sigma$. We distinguish two cases.
    - Case $p = \epsilon$. Then we have $P = \{\epsilon\}$ and $\gamma_k[\hat{\gamma}'_i]_p = \gamma_{2k}$. Furthermore, $\Gamma \Rightarrow \langle u, v\rangle$ is an overlay critical pair, and hence, we have $\Gamma \vdash_{\mathcal{R}} u \mathrel{-\!\!\|\!\!\!\rightarrow}$

$\circ \mathrel{\leftarrow\!\!\!\!\parallel\!\!\!\!-}^* v$ by the parallel-closed assumption. Thus, $u\sigma \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow} \circ \mathrel{\leftarrow\!\!\!\!\parallel\!\!\!\!-}^* v\sigma$ follows from $\vdash_{\mathcal{R}} \Gamma\sigma$ by Definition. Hence we have $\gamma_{1k} = v\sigma \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow}^* \circ \mathrel{\leftarrow\!\!\!\!\parallel\!\!\!\!-} u\sigma = \gamma_k[\hat{\gamma}_i'] = \gamma_{2k}$.

  * Case $p \neq \epsilon$. Then, $\Gamma \Rightarrow \langle u, v\rangle$ is an inner-outer critical pair. Hence, we have $\Gamma \vdash_{\mathcal{R}} u \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow} v$ by the parallel-closed assumption. Thus, $u\sigma \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow} v\sigma$ follows from $\vdash_{\mathcal{R}} \Gamma\sigma$ by Definition. Hence we have $\gamma_{1k} = v\sigma \mathrel{\leftarrow\!\!\!\!\parallel\!\!\!\!-} u\sigma = \gamma_k[\hat{\gamma}_i'] \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow}_{P\setminus\{p\}} \gamma_{2k}$. Now, $Red_{in}(t_1 \mathrel{\leftarrow\!\!\!\!\parallel\!\!\!\!-} \circ \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow} t_2)$ contains $\hat{\gamma}_1, \ldots, \hat{\gamma}_g$. On the other hand, $Red_{in}(\gamma_{1k} \mathrel{\leftarrow\!\!\!\!\parallel\!\!\!\!-} \gamma_k[\hat{\gamma}_i'] \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow} \gamma_{2k})$ contains only subterm occurrences of $\hat{\gamma}_1, \ldots, \hat{\gamma}_{p-1}, \hat{\gamma}_{p+1}, \ldots, \hat{\gamma}_g$. Thus, we have $|Red_{in}(\gamma_{1k} \mathrel{\leftarrow\!\!\!\!\parallel\!\!\!\!-} \gamma_k[\hat{\gamma}_i'] \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow} \gamma_{2k})| < |Red_{in}(t_1 \mathrel{\leftarrow\!\!\!\!\parallel\!\!\!\!-} \circ \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow} t_2)|$. Hence, one can apply the induction hypothesis, to obtain $\gamma_{1k} \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow}^* \circ \mathrel{\leftarrow\!\!\!\!\parallel\!\!\!\!-} \gamma_{2k}$.

- The case $\gamma_k \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow}_P \gamma_{1k}$ and $\gamma_k \mathrel{-\!\!\!\!\parallel\!\!\!\!\rightarrow}_{\{\epsilon\}} \gamma_{2k}$. This case is proved analogously to the previous case. □

Proof (of Theorem 2). Again, the proof is almost same as that of the criteria for TRSs. Let $\mathcal{R}$ be a semi-equational CTRS. We show the claim $t \to t_1$ and $t \to t_2$ imply $t_1 \xrightarrow{*} t_3$ and $t_2 \xrightarrow{=} t_3$ for some t_3. Then the confluence property follows [8]. Let $t \to_{p_i, l_i \to r_i \Leftarrow \Gamma_i} t_i$, α_i be the subterm occurrence at p_i in t, and β_i be the subterm occurrence at p_i in t_i $(i = 1, 2)$. Then we have $\alpha_i \to \beta_i$ for $i = 1, 2$. We distinguish three cases.

- Case $p_1 \parallel p_2$. Then we have $t = C[\alpha_1, \alpha_2]_{p_1,p_2}$ $t_1 = C[\beta_1, \alpha_2]_{p_1,p_2}$, and $t_2 = C[\alpha_1, \beta_2]_{p_1,p_2}$ for some context C. Thus, $t_1 \to_{p_2} C[\beta_1, \beta_2]_{p_1,p_2}\ {}_{p_1}\!\leftarrow t_2$.
- Case $p_1 \geq p_2$. Let $p = p_1 \setminus p_2$. Then we have $r_1\theta \ {}_{\epsilon, l_1 \to r_1 \Leftarrow \Gamma_1}\!\leftarrow l_1\theta \to t_2|_{p_1}$. Suppose there exists $x \in \mathcal{V}(l_1)$ such that the redex occurrence α_2 is contained in a subterm occurrence of $\theta(x)$ in t. Then, by Lemma 4, $t_1 \xrightarrow{=} \circ \leftarrow t_2$. Otherwise the redex occurrence α_1 is not contained in any subterm occurrence of $\theta(x)$ $(x \in \mathcal{V}(l_1))$ in t. Thus, by Lemma 5, there exist a CCP $\Sigma \Rightarrow \langle u, v\rangle$ and a substitution σ, such that $r_2\theta = v\sigma$, $t_2|_{p_1} = u\sigma$ and $\vdash_{\mathcal{R}} \Sigma\sigma$. Thus, by the assumption $t_1 \xrightarrow{*} \circ \xleftarrow{=} t_2$.
- Case $p_1 < p_2$. Let $p = p_2 \setminus p_1$. Then we have $t_1|_{p_2} \leftarrow l_2\theta \to_{\epsilon, l_2 \to r_2 \Leftarrow \Gamma_2} r_2\theta$. Suppose there exists $x \in \mathcal{V}(l_2)$ such that the redex occurrence α_1 is contained in a subterm occurrence of $\theta(x)$ in t. Then, by Lemma 4, $t_1 \to \circ \xleftarrow{=} t_2$. Otherwise the redex occurrence α_1 is not contained in any subterm occurrence of $\theta(x)$ $(x \in \mathcal{V}(l_1))$ in t. Thus, by Lemma 5, there exist a CCP $\Sigma \Rightarrow \langle u, v\rangle$ and a substitution σ, such that $r_2\theta = u\sigma$, $t_1|_{p_2} = v\sigma$ and $\vdash_{\mathcal{R}} \Sigma\sigma$. Thus, by the assumption, $t_1 \xrightarrow{*} \circ \xleftarrow{=} t_2$. □

Proof (of Theorem 4). We here supplement the proof of Theorem 4. For (c), take $S_{l \to r \Leftarrow c}(s, t) = \{\sigma \mid C[l\sigma] = s, C[l\sigma] = t\}$ and then $\mathrm{RED}_1(\Gamma, s, t) = \bigcup_{l \to r \Leftarrow c \in \mathcal{R}} \{\Sigma \mid \langle \Sigma, \mathsf{rhs}(c\sigma)\rangle \in \mathrm{SIM}_0(\Gamma, \mathsf{lhs}(c\sigma)), \sigma \in S_{l \to r \Leftarrow c}(s, t)\}$, where $\mathsf{lhs}(u_1\sigma \approx v_1\sigma, \ldots, u_n\sigma \approx v_n\sigma) = \langle u_1\sigma, \ldots, u_n\sigma\rangle$ and $\mathsf{rhs}(u_1\sigma \approx v_1\sigma, \ldots, u_n\sigma \approx v_n\sigma) = \langle v_1\sigma, \ldots, v_n\sigma\rangle$. For (d), take $A = \bigcup_{(\Psi, s') \in \mathrm{SIM}_0(\Gamma, s)} \{\langle \Gamma', s', t'\rangle \mid (\Gamma', t') \in \mathrm{SIM}_0(\Psi, t)\}$ and $\bigcup\{\mathrm{RED}_1(\Gamma', s', t') \mid \langle \Gamma', s', t'\rangle \in A\}$. For (g), as $\sim_1 = \sim_0 \circ \leftrightarrow_1 \circ \sim_0$, take $\mathrm{SRS}_{010}(\Gamma, s, t) \cup \mathrm{SRS}_{010}(\Gamma, t, s)$.

Now, the condition (i) is equivalent to $\langle \Sigma, t\rangle \in \mathrm{SIM}_0(\Gamma, s)$ for some Σ or $\mathrm{SIM}_1(\Gamma, s, t) \neq \emptyset$. The condition (ii) is equivalent to $\mathrm{RED}_2(\Gamma, s, t) \cup \mathsf{RED}_2(\Gamma, t, s) \neq \emptyset$. The first part of condition (iii) is equivalent to (a) $\Gamma \Vdash_{\mathcal{R}} s \to_2 \circ \sim_0 t$ or (b) $\Gamma \Vdash_{\mathcal{R}} s \to_1 \circ \sim_1 t$ or (c) $\Gamma \Vdash_{\mathcal{R}} s \to_1 \circ \sim_0 t$. (a,c) is equivalent to $\mathrm{RED}_1(\Sigma, s, t') \cup \mathrm{RED}_2(\Sigma, s, t') \neq \emptyset$ for some $\langle \Sigma, t'\rangle \in \mathrm{SIM}_0(\Gamma, t)$. (b) is equivalent to $\mathrm{SIM}_1(\Sigma, s', t) \neq \emptyset$ for some $\langle \Sigma, s'\rangle \in \mathrm{RED}_1(\Gamma, s)$. The second part is similar. $\square$

Proof (of Lemma 1). From (1), $\overset{*}{\leftrightarrow}_0 = \overset{*}{\leftrightarrow}_1$. From $\to_0 \subseteq \to_1$ and (2), $\mathrm{NF}(\to_0) = \mathrm{NF}(\to_1)$. Thus, the claim follows. $\square$

Proof (of Lemma 2). From (1), $\overset{*}{\leftrightarrow}_0 = \overset{*}{\leftrightarrow}_1$. Thus, the claim follows from (2). $\square$

B Comparison to our Definition 3 and Definition 9 of [21] and a proof of Theorem 3

The following definition is obtained by adding the rule **(refl)** to the Definition 9 of [21].

Definition 7. *Let $\mathcal{R}$ be a non-duplicating LR-separated CTRS. Let Γ be a multiset of equations $t' \approx s'$ and a fresh constant $\bullet$. Then relations $t \underset{\Gamma}{\sim} s$ and $t \underset{\Gamma}{\sim\!\triangleright} s$ on terms are inductively defined as follows:*

(asp) $t \underset{\{t \approx s\}}{\sim} s$.
(refl) $t \underset{\{\}}{\sim} t$.
(sym) *If* $t \underset{\Gamma}{\sim} s$ *then* $s \underset{\Gamma}{\sim} t$.
(trans) *If* $t \underset{\Gamma}{\sim} r$ *and* $r \underset{\Gamma'}{\sim} s$ *then* $t \underset{\Gamma \sqcup \Gamma'}{\sim} s$.
(cntxt) *If* $t \underset{\Gamma}{\sim} s$ *then* $C[t] \underset{\Gamma}{\sim} C[s]$.
(rule) *If* $l \to r \Leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n \in \mathcal{R}$ *and* $x_1\theta \underset{\Gamma_i}{\sim} y_i\theta$ $(i = 1, \ldots, n)$ *then* $C[l\theta] \underset{\Gamma}{\sim\!\triangleright} C[r\theta]$ *where* $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_n$.
(bullet) *If* $t \underset{\Gamma}{\sim\!\triangleright} s$ *then* $t \underset{\Gamma \sqcup \{\bullet\}}{\sim} s$.

Note $t \underset{\Gamma}{\sim} s$ in the sense of Definition 9 of [21] implies $t \underset{\Gamma}{\sim} s$ in the sense of Definition 7. On the other hand, $t \underset{\Gamma}{\sim} s$ in the sense of Definition 7 uses **(refl)** rule in the derivation, then $t \underset{\Gamma}{\sim} s$ in the sense of Definition 9 of [21] does not hold.

Now, Lemma 3 of [21] also follows for our Definition of $\underset{\Gamma}{\sim}$ and $\underset{\Gamma}{\sim\!\triangleright}$, since the claim holds for the **(refl)** case trivially.

Lemma 6 (Lemma 3 of [21], generalized). *Let $\Gamma = \{p_1 \approx q_1, \ldots, p_m \approx q_m, \bullet, \ldots, \bullet\}$ be a multiset in which $\bullet$ occurs k times ($k \geq 0$), and let $\mathcal{P}_i : p_i\theta \overset{*}{\leftrightarrow} q_i\theta$ $(i = 1, \ldots, m)$. (1) If $t \underset{\Gamma}{\sim} s$ then there exists a proof $\mathcal{Q} : t\theta \overset{*}{\leftrightarrow} s\theta$ with*

$w(\mathcal{Q}) \leq \Sigma_{i=1}^{m} + k$ *(2) If* $t \underset{\Gamma}{\rightsquigarrow\!\!\triangleright} s$ *then there exists a proof* $\mathcal{Q} : t\theta \to s\theta$ *with* $w(\mathcal{Q}) \leq \Sigma_{i=1}^{m} + k + 1$.

Thus, Theorem 1 of [21] follows for our Definition of $\underset{\Gamma}{\sim}$ and $\underset{\Gamma}{\rightsquigarrow\!\!\triangleright}$.

Theorem 9 (Theorem 1 of [21], generalized). *Let* $\mathcal{R}$ *be a semi-equational non-duplicating LR-separated CTRS. Then* $\mathcal{R}$ *is weight decreasing joinable if for any critical pair* $\Gamma \vdash \langle s, t\rangle$ *of* $\mathcal{R}$*, either (i)* $s \underset{\Sigma}{\sim} t$ *for some* $\Sigma \sqsubseteq \Gamma \sqcup \{\bullet\}$*, (ii)* $s \underset{\Sigma}{\rightsquigarrow\!\!\triangleright} t$ *or* $t \underset{\Sigma}{\rightsquigarrow\!\!\triangleright} s$ *for some* $\Sigma \sqsubseteq \Gamma \sqcup \{\bullet\}$*, or (iii)* $s \underset{\Sigma_1}{\rightsquigarrow\!\!\triangleright} \circ \underset{\Sigma_2}{\sim} t$ *and* $t \underset{\Sigma_1'}{\rightsquigarrow\!\!\triangleright} \circ \underset{\Sigma_2'}{\sim} s$ *for some* $\Sigma_1, \Sigma_2, \Sigma_1', \Sigma_2'$ *such that* $\Sigma_1 \sqcup \Sigma_2 \sqsubseteq \Gamma \sqcup \{\bullet\}$ *and* $\Sigma_1' \sqcup \Sigma_2' \sqsubseteq \Gamma \sqcup \{\bullet\}$.

Below, we abbreviate $\{\overbrace{\bullet, \ldots, \bullet}^{k\text{-times}}\}$ as $\{\bullet^k\}$.

Lemma 7. *Let* Λ *be a multiset of equations. (i) If* $\Lambda \Vdash_{\mathcal{R}} u \sim_k v$ *then* $u \underset{\Delta}{\sim} v$ *for some* $\Delta = \Lambda' \sqcup \{\bullet^k\}$ *such that* $\Lambda' \sqsubseteq \Lambda$*. (ii) If* $\Lambda \Vdash_{\mathcal{R}} u \to_k v$ *then* $u \underset{\Delta}{\rightsquigarrow\!\!\triangleright} v$ *for some* $\Delta = \Lambda' \sqcup \{\bullet^{k-1}\}$ *such that* $\Lambda' \sqsubseteq \Lambda$*. (iii) If* $\Lambda \Vdash_{\mathcal{R}} \langle u_1, \ldots, u_n\rangle \sim_k \langle v_1, \ldots, v_n\rangle$ *then* $u_j \underset{\Delta_j}{\sim} v_j$ $(j = 1, \ldots, n)$ *for some* $\Delta_1, \ldots, \Delta_n$ *such that* $\bigsqcup_j \Delta_j = \Lambda' \sqcup \{\bullet^k\}$ *for some* $\Lambda' \sqsubseteq \Lambda$.

Proof. The proofs of (i)–(iii) proceed by induction on the derivation simultaneously. □

For any multiset Δ of equations and $\bullet$, let $\Delta^\bullet$ be the multiset of $\bullet$ obtained from Δ by removing all equations, and Δ^{eq} be the multiset of equations obtained from Δ by removing all $\bullet$. Furthermore, we denote $|\Delta|$ the length of Δ.

Lemma 8. *Let* Δ *be a multiset of equations and* $\bullet$*. (i) If* $u \underset{\Delta}{\sim} v$ *then* $\Lambda \Vdash_{\mathcal{R}} u \sim_k v$ *for any* $\Lambda \sqsupseteq \Delta^{eq}$*, where* $k = |\Delta^\bullet|$*. (ii) If* $u \underset{\Delta}{\rightsquigarrow\!\!\triangleright} v$ *then* $\Lambda \Vdash_{\mathcal{R}} u \sim_k v$ *for any* $\Lambda \sqsupseteq \Delta^{eq}$*, where* $k = |\Delta^\bullet| + 1$ *(iii) If* $u_j \underset{\Delta_j}{\sim} v_j$ $(j = 1, \ldots, n)$*, then* $\Lambda \Vdash_{\mathcal{R}} \langle u_1, \ldots, u_n\rangle \sim_k \langle v_1, \ldots, v_n\rangle$ *for any* $\Lambda \sqsupseteq \bigsqcup_j \Delta_j^{eq}$*, where* $k = |\bigsqcup_j \Delta_j^\bullet|$.

Proof. The proofs of (i)–(iii) proceed by induction on the derivation simultaneously. □

Lemma 9. *Let* Γ *be a multiset of equations. (i)* $s \underset{\Sigma}{\sim} t$ *for some* $\Sigma \sqsubseteq \Gamma \sqcup \{\bullet\}$ *iff* $\Gamma \Vdash_{\mathcal{R}} s \sim_{\leq 1} t$*. (ii)* $s \underset{\Sigma}{\rightsquigarrow\!\!\triangleright} t$ *for some* $\Sigma \sqsubseteq \Gamma \sqcup \{\bullet\}$ *iff* $\Gamma \Vdash_{\mathcal{R}} s \to_1 t$ *or* $\Gamma \Vdash_{\mathcal{R}} s \to_2 t$*. (iii)* $s \underset{\Sigma_1}{\rightsquigarrow\!\!\triangleright} \circ \underset{\Sigma_2}{\sim} t$ *for some* Σ_1, Σ_2 *such that* $\Sigma_1 \sqcup \Sigma_2 \sqsubseteq \Gamma \sqcup \{\bullet\}$ *iff* $\Gamma \Vdash_{\mathcal{R}} s \to_i \circ \sim_j t$ *with* $i + j \leq 2$.

Proof. (i) ($\Rightarrow$) Suppose $s \underset{\Sigma}{\sim} t$ for some $\Sigma \sqsubseteq \Gamma \sqcup \{\bullet\}$. Then by Lemma 8, $\Lambda \Vdash_{\mathcal{R}} s \sim_k t$ for any $\Lambda \sqsupseteq \Sigma^{eq}$, where $k = |\Sigma^\bullet|$. If $\Sigma \sqsubseteq \Gamma$ then $\bullet \notin \Sigma$, and hence, $\Lambda \Vdash_{\mathcal{R}} s \sim_0 t$ for any $\Lambda \sqsupseteq \Sigma^{eq} = \Sigma$, as $k = |\Sigma^\bullet| = 0$. Thus, $\Lambda \Vdash_{\mathcal{R}} s \sim_0 t$

for any $\Lambda \sqsupseteq \Sigma$. Hence $\Gamma \Vdash_{\mathcal{R}} s \sim_0 t$. Otherwise, we have $\bullet \in \Sigma$, and hence, $\Sigma = \Sigma' \sqcup \{\bullet\}$ for some $\Sigma' \sqsubseteq \Gamma$. Then, $\Lambda \Vdash_{\mathcal{R}} s \sim_1 t$ for any $\Lambda \sqsupseteq \Sigma^{eq} = \Sigma'$, as $k = |\Sigma^{\bullet}| = 1$. Thus, $\Gamma \Vdash_{\mathcal{R}} s \sim_1 t$. Therefore, $\Gamma \Vdash_{\mathcal{R}} s \sim_{\leq 1} t$ holds. ($\Leftarrow$) Firstly, suppose $\Gamma \Vdash_{\mathcal{R}} s \sim_0 t$. Then, by Lemma 7, $s \underset{\Sigma}{\sim} t$ for some $\Sigma = \Gamma' \sqcup \{\bullet^0\}$ such that $\Gamma' \sqsubseteq \Gamma$, i.e. $s \underset{\Sigma}{\sim} t$ for some $\Sigma \sqsubseteq \Gamma$. Next, suppose $\Gamma \Vdash_{\mathcal{R}} s \sim_1 t$. Then, by Lemma 7, $s \underset{\Sigma}{\sim} t$ for some $\Sigma = \Gamma' \sqcup \{\bullet^1\}$ such that $\Gamma' \sqsubseteq \Gamma$, i.e. $s \underset{\Sigma}{\sim} t$ for some $\Sigma \sqsubseteq \Gamma \sqcup \{\bullet\}$. Thus, the claim holds.
(ii) ($\Rightarrow$) Suppose $s \underset{\Sigma}{\rightsquigarrow} t$ for some $\Sigma \sqsubseteq \Gamma \sqcup \{\bullet\}$. Then by Lemma 8, $\Lambda \Vdash_{\mathcal{R}} s \to_k t$ for any $\Lambda \sqsupseteq \Sigma^{eq}$, where $k = |\Sigma^{\bullet}| + 1$. If $\Sigma \sqsubseteq \Gamma$ then $\bullet \notin \Sigma$, and hence, $\Lambda \Vdash_{\mathcal{R}} s \to_1 t$ for any $\Lambda \sqsupseteq \Sigma^{eq} = \Sigma$, as $k = |\Sigma^{\bullet}| + 1 = 1$. Thus, $\Lambda \Vdash_{\mathcal{R}} s \to_1 t$ for any $\Lambda \sqsupseteq \Sigma$. Hence $\Gamma \Vdash_{\mathcal{R}} s \to_1 t$. Otherwise, we have $\bullet \in \Sigma$, and hence, $\Sigma = \Sigma' \sqcup \{\bullet\}$ for some $\Sigma' \sqsubseteq \Gamma$. Then, $\Lambda \Vdash_{\mathcal{R}} s \to_2 t$ for any $\Lambda \sqsupseteq \Sigma^{eq} = \Sigma'$, as $k = |\Sigma^{\bullet}| + 1 = 2$. Thus, $\Gamma \Vdash_{\mathcal{R}} s \to_2 t$. Therefore, $\Gamma \Vdash_{\mathcal{R}} s \to_1 t$ or $\Gamma \Vdash_{\mathcal{R}} s \to_2 t$ holds. ($\Leftarrow$) Firstly, suppose $\Gamma \Vdash_{\mathcal{R}} s \to_1 t$. Then, by Lemma 7, $s \underset{\Sigma}{\rightsquigarrow} t$ for some $\Sigma = \Gamma' \sqcup \{\bullet^0\}$ such that $\Gamma' \sqsubseteq \Gamma$, i.e. $s \underset{\Sigma}{\rightsquigarrow} t$ for some $\Sigma \sqsubseteq \Gamma$. Next, suppose $\Gamma \Vdash_{\mathcal{R}} s \to_2 t$. Then, by Lemma 7, $s \underset{\Sigma}{\rightsquigarrow} t$ for some $\Sigma = \Gamma' \sqcup \{\bullet^1\}$ such that $\Gamma' \sqsubseteq \Gamma$, i.e. $s \underset{\Sigma}{\rightsquigarrow} t$ for some $\Sigma \sqsubseteq \Gamma \sqcup \{\bullet\}$. Thus, the claim holds.
(iii) ($\Rightarrow$) Suppose $s \underset{\Sigma_1}{\rightsquigarrow} \circ \underset{\Sigma_2}{\sim} t$ for some Σ_1, Σ_2 such that $\Sigma_1 \sqcup \Sigma_2 \sqsubseteq \Gamma \sqcup \{\bullet\}$. Firstly, if $\Sigma_1 \sqcup \Sigma_2 \sqsubseteq \Gamma$, then, as in the proof of (i) and (ii), it follows $\Gamma \Vdash_{\mathcal{R}} s \to_1 \circ \sim_0 t$. Secondly, if $\bullet \in \Sigma_1$, then as in the proof of (i) and (ii), it follows $\Gamma \Vdash_{\mathcal{R}} s \to_2 \circ \sim_0 t$. Finally, if $\bullet \in \Sigma_2$, then as in the proof of (i) and (ii), it follows $\Gamma \Vdash_{\mathcal{R}} s \to_1 \circ \sim_1 t$. Thus, in any case, $\Gamma \Vdash_{\mathcal{R}} s \to_i \circ \sim_j t$ with $i + j \leq 2$. ($\Leftarrow$) Suppose $\Gamma \Vdash_{\mathcal{R}} s \to_i \circ \sim_j t$ with $i + j \leq 2$. Then we have cases (a) $\Gamma \Vdash_{\mathcal{R}} s \to_1 u \sim_0 t$, (b) $\Gamma \Vdash_{\mathcal{R}} s \to_1 u \sim_1 t$, and (c) $\Gamma \Vdash_{\mathcal{R}} s \to_2 u \sim_0 t$. In case (a), there exist Γ_1, Γ_2 such that $\Gamma = \Gamma_1 \sqcup \Gamma_2$, $\Gamma_1 \Vdash_{\mathcal{R}} s \to_1 u$ and $\Gamma_2 \Vdash_{\mathcal{R}} u \sim_0 t$. Then, as in the proof of (i) and (ii), $s \underset{\Sigma_1}{\rightsquigarrow} u$ for some for some $\Sigma_1 \sqsubseteq \Gamma_1$ and $u \underset{\Sigma_2}{\sim} t$ for some for some $\Sigma_2 \sqsubseteq \Gamma_2$. In case (b), similarly, we have $s \underset{\Sigma_1}{\rightsquigarrow} u$ for some for some $\Sigma_1 \sqsubseteq \Gamma_1$ and $u \underset{\Sigma_2}{\sim} t$ for some for some $\Sigma_2 \sqsubseteq \Gamma_2 \sqcup \{\bullet\}$. In case (c), similarly, we have $s \underset{\Sigma_1}{\rightsquigarrow} u$ for some for some $\Sigma_1 \sqsubseteq \Gamma_1 \sqcup \{\bullet\}$. and $u \underset{\Sigma_2}{\sim} t$ for some for some $\Sigma_2 \sqsubseteq \Gamma_2$. Thus, the claim holds. □

Proof (of Theorem 3). It follows immediately from Lemma 9, by noting $\Gamma \Vdash_{\mathcal{R}} s \to_1 t$ implies $\Gamma \Vdash_{\mathcal{R}} s \sim_1 t$. □

C Some detailed proofs

Proof (of Lemma 7). We prove (i)–(iii) simultaneously by induction on the derivation.

1. Case $\Gamma \sqcup \{u \approx v\} \Vdash_{\mathcal{R}} u \sim_0 v$. The claim holds since $u \underset{\{u \approx v\}}{\sim} v$ by **asp**.

2. Case $\Gamma \Vdash_{\mathcal{R}} t \sim_0 t$ The claim holds since $t \underset{\{\}}{\sim} t$ by **refl**.
3. Case $\Gamma \Vdash_{\mathcal{R}} s \sim_i t$ is derived from $\Gamma \Vdash_{\mathcal{R}} t \sim_i s$. By induction hypothesis, $t \underset{\Delta}{\sim} s$ for some $\Delta = \Gamma' \sqcup \{\bullet^i\}$ such that $\Gamma' \sqsubseteq \Gamma$. Then $s \underset{\Delta}{\sim} t$ by **sym**, and the claim holds.
4. Case $\Gamma \sqcup \Sigma \Vdash_{\mathcal{R}} s \sim_{i+j} u$ is derived from $\Gamma \Vdash_{\mathcal{R}} s \sim_i t$ and $\Sigma \Vdash_{\mathcal{R}} t \sim_j u$ By induction hypothesis, $s \underset{\Delta_1}{\sim} t$ for some $\Delta_1 = \Gamma' \sqcup \{\bullet^i\}$ such that $\Gamma' \sqsubseteq \Gamma$, and $t \underset{\Delta_2}{\sim} u$ for some $\Delta_2 = \Sigma' \sqcup \{\bullet^j\}$ such that $\Sigma' \sqsubseteq \Sigma$. Take $\Delta = \Delta_1 \sqcup \Delta_2$. Then $s \underset{\Delta}{\sim} u$ by **trans**. Furthermore, $\Delta = \Delta_1 \sqcup \Delta_2 = \Gamma' \sqcup \{\bullet^i\} \sqcup \Sigma' \sqcup \{\bullet^j\} = \Gamma' \sqcup \Sigma' \sqcup \{\bullet^{i+j}\}$, and $\Gamma' \sqcup \Sigma' \sqsubseteq \Gamma \sqcup \Sigma$. Hence the claim holds.
5. Case $\Gamma \Vdash_{\mathcal{R}} C[s] \sim_i C[t]$ is derived from $\Gamma \Vdash_{\mathcal{R}} s \sim_i t$ By induction hypothesis, $s \underset{\Delta}{\sim} t$ for some $\Delta = \Gamma' \sqcup \{\bullet^i\}$ such that $\Gamma' \sqsubseteq \Gamma$. Then $C[s] \underset{\Delta}{\sim} C[t]$ by **cntxt**, and the claim holds.
6. Case $\bigsqcup_j \Gamma_j \Vdash_{\mathcal{R}} \langle u_1, \ldots, u_n \rangle \sim_k \langle v_1, \ldots, v_n \rangle$ is derived from $\Gamma_1 \Vdash_{\mathcal{R}} u_1 \sim_{i_1} v_1, \ldots, \Gamma_n \Vdash_{\mathcal{R}} u_n \sim_{i_n} v_n$ where $k = \sum_j i_j$. By induction hypothesis, for each $j = 1, \ldots, n$, $u_j \underset{\Delta_j}{\sim} v_j$ for some $\Delta_j = \Gamma'_j \sqcup \{\bullet^{i_j}\}$ such that $\Gamma'_j \sqsubseteq \Gamma_j$. Since $\bigsqcup_j \Delta_j = \bigsqcup_j \Gamma'_j \sqcup \{\bullet^k\}$ and $\bigsqcup_j \Gamma'_j \sqsubseteq \bigsqcup_j \Gamma_j$, the claim holds.
7. Case $\Gamma \Vdash_{\mathcal{R}} s \sim_i t$ is derived from $\Gamma \Vdash_{\mathcal{R}} s \to_i t$. By induction hypothesis, $s \underset{\Delta}{\sim\!\!\triangleright} t$ for some $\Delta = \Gamma' \sqcup \{\bullet^{i-1}\}$ such that $\Gamma' \sqsubseteq \Gamma$. Then $s \underset{\Delta \sqcup \{\bullet\}}{\sim} t$ by **bullet** and $\Delta \sqcup \{\bullet\} = \Gamma' \sqcup \{\bullet^i\}$. Thus, the claim holds.
8. Case $\Gamma \Vdash_{\mathcal{R}} C[l\sigma] \to_{i+1} C[r\sigma]$ is derived from $\Gamma \Vdash_{\mathcal{R}} \langle x_1\sigma, \ldots, x_n\sigma \rangle \sim_i \langle y_1\sigma, \ldots, y_n\sigma \rangle$ where $l \to r \Leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n \in \mathcal{R}$. By induction hypothesis, $x_j\sigma \underset{\Delta_j}{\sim} y_j\sigma$ $(j = 1, \ldots, n)$ for some $\Delta_1, \ldots, \Delta_n$ such that $\bigsqcup_j \Delta_j = \Gamma' \sqcup \{\bullet^i\}$ for some $\Gamma' \sqsubseteq \Gamma$. Then, by **rule**, we have $C[l\theta] \underset{\Delta}{\sim\!\!\triangleright} C[r\theta]$ where $\Delta = \bigsqcup_j \Delta_j$. Thus, the claim holds. □

Proof (of Lemma 8). We prove (i)–(iii) simultaneously by induction on the derivation.

1. Case **(asp)**. We have $t \underset{\{t \approx s\}}{\sim} s$. Then $\Lambda \Vdash_{\mathcal{R}} t \sim_0 s$ for any $\Lambda \sqsupseteq \{t \approx s\}$ by definition.
2. Case **(refl)**. We have $t \underset{\{\}}{\sim} t$. Then $\Lambda \Vdash_{\mathcal{R}} t \sim_0 t$ for any Λ by definition.
3. Case **(sym)**. Suppose $s \underset{\Gamma}{\sim} t$ is derived from $t \underset{\Gamma}{\sim} s$. Let $\Lambda \sqsupseteq \Gamma^{eq}$. Then by induction hypothesis, $\Lambda \Vdash_{\mathcal{R}} t \sim_k s$, where $k = |\Gamma^\bullet|$. Then, it follows $\Lambda \Vdash_{\mathcal{R}} t \sim_k s$ by definition.
4. Case **(trans)**. Suppose $t \underset{\Gamma \sqcup \Sigma}{\sim} s$ is derived from $t \underset{\Gamma}{\sim} r$ and $r \underset{\Sigma}{\sim} s$. Let $\Lambda \sqsupseteq (\Gamma \sqcup \Sigma)^{eq}$. Then, there exist Λ_1, Λ_2 such that $\Lambda = \Lambda_1 \sqcup \Lambda_2$, $\Lambda_1 \sqsupseteq \Gamma^{eq}$ and $\Lambda_2 \sqsupseteq \Sigma^{eq}$. Then, by induction hypothesis, $\Lambda_1 \Vdash_{\mathcal{R}} t \sim_{k_1} s$ where $k_1 = |\Gamma^\bullet|$, and $\Lambda_2 \Vdash_{\mathcal{R}} t \sim_{k_2} s$ where $k_2 = |\Sigma^\bullet|$. Then, it follows $\Lambda \Vdash_{\mathcal{R}} t \sim_{k_1+k_2} s$ by definition. As $k_1 + k_2 = |\Gamma^\bullet| + |\Sigma^\bullet| = |(\Gamma \sqcup \Sigma)^\bullet|$, the claim follows.

5. Case **(cntxt)**. Suppose $C[t] \underset{\Gamma}{\sim} C[s]$ is derived from $t \underset{\Gamma}{\sim} s$. Let $\Lambda \sqsupseteq \Gamma^{eq}$. Then by induction hypothesis, $\Lambda \Vdash_{\mathcal{R}} t \sim_k s$, where $k = |\Gamma^\bullet|$. Then, it follows $\Lambda \Vdash_{\mathcal{R}} C[t] \sim_k C[s]$ by definition.
6. Case **(rule)**. Suppose $C[l\theta] \underset{\Gamma}{\sim\!\rhd} C[r\theta]$ is derived from $x_1\theta \underset{\Gamma_i}{\sim} y_i\theta$ $(i = 1, \ldots, n)$, where $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_n$ and $l \to r \Leftarrow x_1 \approx y_1, \ldots, x_n \approx y_n \in \mathcal{R}$. Let $\Lambda \sqsupseteq \Gamma^{eq}$. Then, there exist $\Lambda_1, \ldots, \Lambda_n$ such that $\Lambda = \bigsqcup_j \Lambda_j$ and $\Lambda_j \sqsupseteq \Gamma_j^{eq}$ for each $1 \leq j \leq n$. Hence, by induction hypothesis, $\Lambda_j \Vdash_{\mathcal{R}} x_j\theta \sim_{k_j} y_j\theta$ where $k_j = |\Gamma_j^\bullet|$ for each $1 \leq j \leq n$. Then, by definition, $\Lambda \Vdash_{\mathcal{R}} \langle x_1\theta, \ldots, x_n\theta\rangle \sim_{k'} \langle y_1\theta, \ldots, y_n\theta\rangle$ where $k' = \sum_j k_j = \sum_j |\Gamma_j^\bullet| = |(\bigsqcup_j \Gamma_j)^\bullet| = |\Gamma^\bullet|$. Then, by definition, $\Lambda \Vdash_{\mathcal{R}} C[l\theta] \sim_{k'+1} C[r\theta]$.
7. Case **(bullet)**. Suppose $s \underset{\Gamma \sqcup \{\bullet\}}{\sim} t$ is derived from $t \underset{\Gamma}{\sim\!\rhd} s$. Let $\Lambda \sqsupseteq \Gamma^{eq}$. Then by induction hypothesis, $\Lambda \Vdash_{\mathcal{R}} s \sim\!\rhd_k s$, where $k = |\Gamma^\bullet| + 1$. Then, it follows $\Lambda \Vdash_{\mathcal{R}} t \sim_k s$ by definition. □

D Additional examples and experiments on presented examples

We here presents an additional example and the result of experiments on examples in the paper.

Example 8. Let

$$\mathcal{R} = \left\{ \begin{array}{lll} f(x,x) & \to a & c \;\;\;\;\; \to h(c, g(c)) \\ h(x, g(x)) & \to f(c, h(x, g(c))) & k(c) \to k(h(h(c, g(c)), g(c))) \end{array} \right\}.$$

In the Table 3, we show the result of experiments for the examples presented in the paper. Here, $\mathcal{R}_n$ show the TRS $\mathcal{R}$ in Example n. ✓ shows success and × shows failure. The experiment is performed on the same PC mentioned in Section 6, with 60 sec. of timeout. The columns below the title ACP(direct) show the results of ACP without using the decomposition methods. For Example $\mathcal{R}_2$, only Corollary 1 is effective. For Example $\mathcal{R}_3$, only Corollary 2 is effective. Furthermore, CSI and FORT fail for all examples.

without (rev)	(ω)	**(pcl)**	**(scl)**	**(wd)**	**(sc)** 1/2/3	**(dc)** 1/2/3	**(rr)**	**(cp)**
$\mathcal{R}_2$	×	✓	×	×	×/×/×	×/×/×	×	×
$\mathcal{R}_3$	×	×	✓	×	×/×/×	×/×/×	×	×
$\mathcal{R}_5$	×	✓	✓	✓	×/×/×	×/×/×	×	×
$\mathcal{R}_6$	×	×	×	×	×/✓/✓	×/✓/✓	×	×
$\mathcal{R}_7$	×	×	×	×	×/×/×	×/×/×	×	×
$\mathcal{R}_8$	×	✓	✓	✓	×/×/×	×/×/×	×	×

with (rev)	(ω)	**(pcl)**	**(scl)**	**(wd)**	**(sc)** 1/2/3	**(dc)** 1/2/3	**(rr)**	**(cp)**
$\mathcal{R}_2$	×	✓	×	×	×/×/×	×/×/×	×	×
$\mathcal{R}_3$	×	×	×	×	×/×/×	×/×/×	×	×
$\mathcal{R}_5$	×	×	×	×	×/×/×	×/×/×	×	×
$\mathcal{R}_6$	×	×	×	×	×/✓/✓	×/✓/✓	×	×
$\mathcal{R}_7$	×	×	×	×	×/×/×	×/×/×	×	×
$\mathcal{R}_8$	×	×	×	×	×/×/×	×/×/×	×	×

	ACP	ACP(direct)	CSI	FORT
$\mathcal{R}_2$	✓	✓	×	×
$\mathcal{R}_3$	✓	✓	timeout	×
$\mathcal{R}_5$	✓	✓	×	×
$\mathcal{R}_6$	✓	✓	×	×
$\mathcal{R}_7$	✓	timeout	timeout	×
$\mathcal{R}_8$	✓	✓	timeout	×

Table 3. Test for presented examples