Proving Confluence of Term Rewriting Systems via Persistency and Decreasing Diagrams

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Abstract. The decreasing diagrams technique (van Oostrom, 1994) has been successfully used to prove confluence of rewrite systems in various ways; using rule-labelling (van Oostrom, 2008), it can also be applied directly to prove confluence of some linear term rewriting systems (TRSs) automatically. Some efforts for extending the rule-labelling are known, but non-left-linear TRSs are left beyond the scope. Two methods for automatically proving confluence of non-(left-)linear TRSs with the rule-labelling are given. The key idea of our methods is to combine the decreasing diagrams technique with persistency of confluence (Aoto & Toyama, 1997).

Keywords: Confluence, Persistency, Decreasing Diagrams, Rule-Labelling, Non-Linear, Term Rewriting Systems

1 Introduction

Decreasing diagrams [11] give a characterization of confluence of abstract rewrite systems; the criterion based on decreasing diagrams can be adapted to prove confluence of rewrite systems in various ways. In particular, *rule-labelling* [12] has been adapted to prove confluence of *left-linear* TRSs [1, 8, 19] automatically. A property of TRSs is said to be *persistent* if the property is preserved under elimination of sorts [20]. It is shown in [2] that confluence is persistent, that is, if a many-sorted TRS is confluent on (many-sorted) terms then so is the underlying unsorted TRS on all (i.e. including ill-sorted) terms.

In this paper, the decreasing diagrams technique and persistency of confluence are combined to give methods for proving confluence of *non-linear* TRSs automatically. For proving confluence of TRSs $\mathcal{R}$, we consider a subsystem $\mathcal{R}^\tau_{nl}$ which is obtained from some many-sorted version $\mathcal{R}^\tau$ of $\mathcal{R}$. Based on assumptions on the subsystem $\mathcal{R}^\tau_{nl}$, we develop two confluence criteria based on decreasing diagrams with rule-labelling—one of the criteria is based on the assumption that $\mathcal{R}^\tau_{nl}$ is terminating, and the other is based on the assumption that $\mathcal{R}^\tau_{nl}$ is innermost normalizing. These two criteria are incomparable, and the proofs of the correctness are given independently. Both of the criteria, however, can be applied to prove confluence of non-left-linear non-terminating TRSs, for which no decreasing diagrams technique with rule-labelling has been known and only few techniques for proving confluence have been known.

The rest of the paper is organized as follows. Section 2 covers preliminaries; some common notions and notations to be used in Sections 3 and 4 are also presented. In Section 3, we introduce the class of strongly quasi-linear TRSs and show a confluence criterion for TRSs in this class. In Section 4, we introduce the class of quasi-linear TRSs and show a confluence criterion for TRSs in this class. We also show that these two criteria are incomparable. In Section 5, we report on an implementation of these criteria in our confluence prover ACP [3] and on experiments. Related work is also explained in Section 5. Section 6 concludes.

2 Preliminaries

We fix notations assuming basic familiarity with term rewriting [4].

The transitive (reflexive, transitive and reflexive, equivalence) closure of a relation $\to$ (on a set A) is denoted by $\xrightarrow{+}$ ($\xrightarrow{=}$, $\xrightarrow{*}$, $\xleftrightarrow{*}$, respectively). An element a is a *normal form* if $a \to b$ for no b; *normalizing* if $a \xrightarrow{*} b$ for some normal form b; *terminating* if there exists no infinite sequence $a = a_0 \to a_1 \to \cdots$. The relation $\to$ is *normalizing* (*terminating*) if so are all $a \in A$; *confluent* if $\xleftarrow{*} \circ \xrightarrow{*} \subseteq \xrightarrow{*} \circ \xleftarrow{*}$.

We denote a set of (arity-fixed) function symbols by $\mathcal{F}$, an enumerable set of variables by $\mathcal{V}$, and the set of terms by $\mathrm{T}(\mathcal{F},\mathcal{V})$. A variable in a term t is *linear* if it occurs only once in t, otherwise *non-linear*. The set of variables (linear variables, non-linear variables) in t is denoted by $\mathcal{V}(t)$ ($\mathcal{V}_l(t)$, $\mathcal{V}_{nl}(t)$, respectively). A term t is *ground* if $\mathcal{V}(t) = \emptyset$. A *position* is a sequence of positive integers, where ϵ stands for the empty sequence. The set of positions (function positions, variable positions) of a term t is denoted by $\mathrm{Pos}(t)$ ($\mathrm{Pos}_{\mathcal{F}}(t)$, $\mathrm{Pos}_{\mathcal{V}}(t)$, respectively). We use $\leqslant$ for the *prefix order* on positions. Positions p and q are *disjoint* ($p \parallel q$) if $p \nleqslant q$ and $q \nleqslant p$. The symbol (subterm) of a term t at the position p is denoted by $t(p)$ ($t|_p$, respectively). The *subterm relation* is denoted by $\trianglelefteq$; its strict part is by $\triangleleft$. We write $\theta : X \to T$ to if the substitution θ satisfies $\theta(x) = x$ for all $x \in \mathcal{V} \setminus X$ and $\theta(x) \in T$ for any $x \in X$. The *most general unifier* of s and t is denoted by $\mathrm{mgu}(s,t)$. A *rewrite rule* $l \to r$ satisfies $l \notin \mathcal{V}$ and $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. Rewrite rules are identified modulo renaming of variables. A rewrite rule $l \to r$ is *linear* if l and r are linear. The set of non-linear variables of a rewrite rule $l \to r$ is given by $\mathcal{V}_{nl}(l \to r) = \mathcal{V}_{nl}(l) \cup \mathcal{V}_{nl}(r)$; that of linear variables is by $\mathcal{V}_l(l \to r) = \mathcal{V}(l) \setminus \mathcal{V}_{nl}(l \to r)$. A *term rewriting system* (*TRS*) is a set $\mathcal{R}$ of rewrite rules; $\mathcal{R}$ is linear if so are all its rewrite rules. A *rewrite step* $s \to_{\mathcal{R}} t$ is written as $s \to_{p,l\to r,\theta} t$ to specify the position p, the rewrite rule $l \to r \in \mathcal{R}$ and the substitution θ employed. If $s \to_{p,l\to r,\theta} t$ or $s \leftarrow_{p,l\to r,\theta} t$, we (ambiguously) write $s \leftrightarrow_{p,l\to r,\theta} t$. If not necessary, subscripts $p, l \to r, \theta, \mathcal{R}$ will be dropped. The set of normal forms (w.r.t. the *rewrite relation* $\to_{\mathcal{R}}$) is denoted by $\mathrm{NF}_{\mathcal{R}}(\mathcal{F},\mathcal{V})$, or just $\mathrm{NF}(\mathcal{F},\mathcal{V})$. A TRS $\mathcal{R}$ is normalizing (terminating, confluent) if so is $\to_{\mathcal{R}}$. We write $s \to^{im} t$ if $s \to_p t$ is *innermost*, i.e. any proper subterm of $s|_p$ is a normal form. A term or a TRS is *innermost normalizing* (*innermost terminating*) if it is normalizing (terminating, respectively) w.r.t. $\to^{im}$.

A *conversion* $\gamma : s_1 \leftrightarrow_{l_1 \to r_1} s_2 \leftrightarrow_{l_2 \to r_2} \cdots \leftrightarrow_{l_{n-1} \to r_{n-1}} s_n$ is specified as $\gamma : s_1 \overset{*}{\leftrightarrow} s_n$ if the detail is not necessary. We put $\mathit{Rules}(\gamma) = \{l_i \to r_i \mid 1 \leq i < n\}$, or $\mathit{Rules}(s_1 \overset{*}{\leftrightarrow} s_n) = \{l_i \to r_i \mid 1 \leq i < n\}$ for brevity. A conversion $t_1 \leftarrow_{p_1, l_1 \to r_1, \theta_1} s \to_{p_2, l_2 \to r_2, \theta_2} t_2$ is called a *peak*; it is a *disjoint* peak if $p_1 \parallel p_2$, it is a *variable* peak if $p_1 = p_2.o.q$ for some $o \in \mathrm{Pos}_{\mathcal{V}}(l_2)$ and q or the other way round, it is a *overlap* peak if $p_1 = p_2.o$ for some $o \in \mathrm{Pos}_{\mathcal{F}}(l_2)$ or the other way round; furthermore, an overlap peak is *trivial* if $p_1 = p_2$ and $l_1 \to r_1 = l_2 \to r_2$. For rewrite rules $l_1 \to r_1, l_2 \to r_2 \in \mathcal{R}$ (w.l.o.g. $\mathcal{V}(l_1) \cap \mathcal{V}(l_2) = \emptyset$), any non-trivial overlap peak of the form $l_2[r_1]_q\theta \leftarrow_{q, l_1 \to r_1, \theta} l_2\theta \to_{\epsilon, l_2 \to r_2, \theta} r_2\theta$ is called a *critical* peak, if $q \in \mathrm{Pos}_{\mathcal{F}}(l_2)$ and $\theta = \mathrm{mgu}(l_1, l_2|_q)$. The set of critical peaks of rules from $\mathcal{R}$ is denoted by $\mathrm{CP}(\mathcal{R})$.

Decreasing Diagrams. Let $\succ$ be a partial order on a set $\mathcal{L}$ of labels. For $\alpha, \beta \in \mathcal{L}$, subsets $\curlyvee\alpha, \curlyvee\alpha\vee\beta \subseteq \mathcal{L}$ are given by $\curlyvee\alpha = \{\gamma \in \mathcal{L} \mid \gamma \prec \alpha\}$ and $\curlyvee\alpha\vee\beta = \{\gamma \in \mathcal{L} \mid \gamma \prec \alpha \vee \gamma \prec \beta\}$. Let A be a set and $\to_\alpha$ be a relation on A for each $\alpha \in \mathcal{L}$. We let $\to_\ell = \bigcup_{\alpha \in \ell} \to_\alpha$ for $\ell \subseteq \mathcal{L}$. Then the relation $\to_{\mathcal{L}}$ is said to be *locally decreasing w.r.t.* $\succ$ if, for any $\alpha, \beta \in \mathcal{L}$, $\leftarrow_\alpha \circ \to_\beta \subseteq \overset{*}{\leftrightarrow}_{\curlyvee\alpha} \circ \overset{=}{\to}_\beta \circ \overset{*}{\leftrightarrow}_{\curlyvee\alpha\vee\beta} \circ \overset{=}{\leftarrow}_\alpha \circ \overset{*}{\leftrightarrow}_{\curlyvee\beta}$.

Proposition 2.1 (Confluence by decreasing diagrams [12]). *A relation $\to_{\mathcal{L}}$ is confluent if it is locally decreasing w.r.t. some well-founded partial order $\succ$ on $\mathcal{L}$.*

In order to apply this proposition for proving the confluence of a TRS $\mathcal{R}$, we need to set relations $\to_\alpha$ ($\alpha \in \mathcal{L}$) on $\mathrm{T}(\mathcal{F}, \mathcal{V})$ such that $\bigcup_{\alpha \in \mathcal{L}} \to_\alpha = \to_{\mathcal{R}}$. For this, we consider a *labelling function*, say *lab*, that assigns a label to each rewrite step, and put $s \to_\alpha t$ if $\alpha = \mathit{lab}(s \to t)$. We say a peak $t_1 \leftarrow_\alpha s \to_\beta t_2$ is *decreasing w.r.t. lab (and* $\succ$*)* if there exists a conversion $t_1 \overset{*}{\leftrightarrow}_{\curlyvee\alpha} \circ \overset{=}{\to}_\beta \circ \overset{*}{\leftrightarrow}_{\curlyvee\alpha\vee\beta} \circ \overset{=}{\leftarrow}_\alpha \circ \overset{*}{\leftrightarrow}_{\curlyvee\beta} t_n$ (Figure 1). Then, by the proposition, $\mathcal{R}$ is confluent if there exist a labelling function *lab* such that any peak is decreasing w.r.t. *lab*.

Persistency. Let $\mathcal{S}$ be a set of *sorts*. A *sort assignment* τ assigns $\tau(x) \in \mathcal{S}$ to each variable $x \in \mathcal{V}$ and $\tau(f) \in \mathcal{S}^{n+1}$ to each function symbol $f \in \mathcal{F}$ of arity n, in such a way that $\{x \in \mathcal{V} \mid \tau(x) = \sigma\}$ is infinite for any $\sigma \in \mathcal{S}$. Sort assignment τ induces a *many-sorted signature*—the set of well-sorted terms is denoted by $\mathrm{T}(\mathcal{F}, \mathcal{V})^\tau$. We write t^τ to denote $t \in \mathrm{T}(\mathcal{F}, \mathcal{V})^\tau$; $\tau(t) = \sigma$ if the sort of $t \in \mathrm{T}(\mathcal{F}, \mathcal{V})^\tau$ is σ. A quasi-order $\gtrsim$ on $\mathcal{S}$ is given like this: $\sigma \gtrsim \rho$ if there exists a well-sorted term of sort σ having a subterm of sort ρ.

A sort assignment τ is *consistent* with a TRS $\mathcal{R}$ if (l and r are well-sorted and) $\tau(l) = \tau(r)$ for all $l \to r \in \mathcal{R}$ where w.l.o.g. the sets of variables in rewrite rules are supposed to be mutually disjoint. A sort assignment τ consistent with a TRS $\mathcal{R}$ induces a *many-sorted TRS* $\mathcal{R}^\tau$; the rewrite relation of $\mathcal{R}^\tau$ (and hence the notions of confluence, etc.) is defined on $\mathrm{T}(\mathcal{F}, \mathcal{V})^\tau$. If no confusion arises, many-sorted TRSs are called TRSs for simplicity.

Proposition 2.2 (Persistency of confluence [2]). *For any sort assignment τ consistent with $\mathcal{R}$, $\mathcal{R}^\tau$ is confluent iff $\mathcal{R}$ is confluent.*

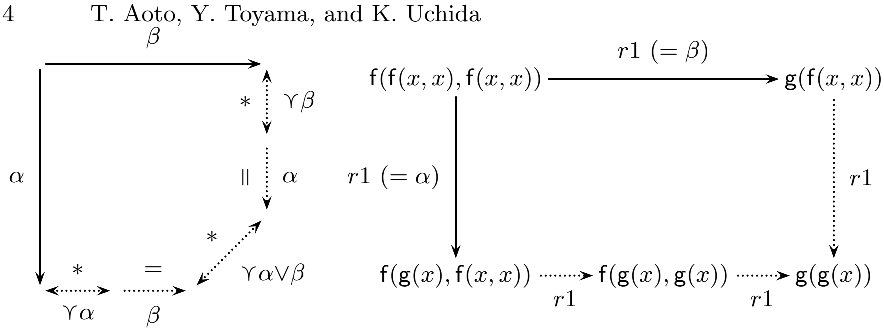

Fig. 1. Decreasing peak and an non-decreasing peak by $r1 : \mathsf{f}(x, x) \to \mathsf{g}(x)$

2.1 Non-Linear Sorts

We will give confluence criteria based on decreasing diagrams in the following two sections, one in each of sections; these two criteria are incomparable and their correctness are proven independently. Some notions, however, are shared—we introduce these notions in this subsection.

Our confluence criteria based on decreasing diagrams aim at dealing with non-linear rewrite rules. For this, we extend the *rule-labelling* [12], which considers a function $\delta : \mathcal{R} \to \mathcal{L}$ and labels each rewrite step by $lab(s \to_{l \to r} t) = \delta(l \to r)$. To show the confluence of a TRS $\mathcal{R}$ by decreasing diagrams with only rule-labelling, $\mathcal{R}$ needs to be linear [12, 1]—if a non-linear rewrite rule is contained then one always obtains non-decreasing peaks (Figure 1). Our idea to deal with such cases is to restrict rewrite rules usable in instantiations of non-linear variables (of other rewrite rules) by considering well-sorted terms. The set of rewrite rules usable in instantiations of non-linear variables of rewrite rules in U^τ is denoted by U^τ_{nl} and is formally given as below.

Definition 2.3 (non-linear sort, many-sorted TRS $\mathcal{R}^\tau_{nl}$). *Let $\mathcal{R}$ be a TRS, τ a sort assignment consistent with $\mathcal{R}$ and $U^\tau \subseteq \mathcal{R}^\tau$. A sort $\sigma \in \mathcal{S}$ is said to be a* non-linear sort *of U^τ if there exist $l \to r \in U^\tau$ and $x \in \mathcal{V}_{nl}(l \to r)$ such that $\tau(x) = \sigma$. The set of non-linear sorts of U^τ is denoted by $\mathcal{S}_{nl}(U^\tau)$. By non-linear sorts, we mean non-linear sorts of $\mathcal{R}^\tau$. We define the set $U^\tau_{nl} \subseteq \mathcal{R}^\tau$ as*

$$U^\tau_{nl} = \{l \to r \in \mathcal{R}^\tau \mid \exists \sigma \in \mathcal{S}_{nl}(U^\tau).\ \tau(l) \lesssim \sigma\}.$$

(U^τ_{nl} is written as $\mathcal{R}^\tau_{nl}$ if we take $U^\tau = \mathcal{R}^\tau$.) We also put $U^\tau_l = \mathcal{R}^\tau \setminus U^\tau_{nl}$.

Clearly, $\mathcal{S}_{nl}$ and $()_{nl}$ are monotone, i.e. $U^\tau \subseteq T^\tau$ implies $\mathcal{S}_{nl}(U^\tau) \subseteq \mathcal{S}_{nl}(T^\tau)$ and $U^\tau_{nl} \subseteq T^\tau_{nl}$.

Example 2.4. Let $\mathcal{S} = \{0, 1, 2\}$ and

$$\mathcal{R} = \left\{ \begin{array}{llll} (r1) & \mathsf{f}(x, x) \to \mathsf{f}(\mathsf{h}(\mathsf{b}), \mathsf{h}(\mathsf{a})) & (r2) & \mathsf{h}(x) \to \mathsf{k}(x, x) \\ (r3) & \mathsf{k}(\mathsf{a}, \mathsf{b}) \to \mathsf{h}(\mathsf{a}) & (r4) & \mathsf{a} \quad\to \mathsf{b} \end{array} \right\}.$$

Take a sort assignment $\tau = \{\mathsf{f} : 1 \times 1 \to 2, \mathsf{k} : 0 \times 0 \to 1, \mathsf{h} : 0 \to 1, \mathsf{a} : 0, \mathsf{b} : 0\}$ consistent with $\mathcal{R}$. We have $\mathcal{S}_{nl}(\{(r1)\}) = \{1\}$, $\mathcal{S}_{nl}(\{(r2)\}) = \{0\}$,

$\mathcal{S}_{nl}(\{(r3)\}) = \mathcal{S}_{nl}(\{(r4)\}) = \emptyset$ and $\mathcal{S}_{nl}(\mathcal{R}^\tau) = \{0,1\}$. We have $\mathcal{R}^\tau_{nl} = \{(r1)\}_{nl} = \{(r2),(r3),(r4)\}$, $\{(r2)\}_{nl} = \{(r4)\}$, $\{(r3)\}_{nl} = \{(r4)\}_{nl} = \emptyset$ and $\mathcal{R}^\tau_l = \{(r1)\}$.

Note that any subterm of a term of non-linear sort has non-linear sort. Similarly, if $s \to_{l\to r} t$ and $\tau(s) \in \mathcal{S}_{nl}(U^\tau)$ then $l \to r \in U^\tau_{nl}$, and thus, there is no critical peak of the form $t_1 \leftarrow_{\mathcal{R}^\tau_l} \circ \to_{\mathcal{R}^\tau_{nl}} t_2$.

In the next section (Section 3), we will give a confluence criterion that can be applied if $\mathcal{R}^\tau_{nl}$ is terminating. In Section 4, we consider the case that $\mathcal{R}^\tau_{nl}$ is (possibly not terminating but) innermost normalizing.

3 Confluence of Strongly Quasi-Linear TRSs

In this section, we give a confluence criterion based on the decreasing diagrams and strong quasi-linearity, a notion for many-sorted TRSs given as follows.

Definition 3.1 (strongly quasi-linear). *A many-sorted TRS $\mathcal{R}^\tau$ is* strongly quasi-linear *if the many-sorted TRS $\mathcal{R}^\tau_{nl}$ is terminating.*

Clearly, if $\mathcal{R}^\tau$ is strongly quasi-linear, any (well-sorted) term of non-linear sort is terminating. Note that any (well-sorted) term is terminating w.r.t. $\mathcal{R}^\tau_{nl}$. For strongly quasi-linear TRSs, the following labelling function is considered.

Definition 3.2 (labelling for strongly quasi-linear TRS). *Let $\mathcal{R}^\tau$ be a strongly quasi-linear TRS.*

1. *Let $\mathcal{L}$ be a set and $>$ a well-founded partial order on it. We consider the set $\mathcal{L} \cup \mathrm{T}(\mathcal{F} \cup \mathcal{V})^\tau$ as the set of labels.*
2. *We define a relation $\succ$ on $\mathcal{L} \cup \mathrm{T}(\mathcal{F} \cup \mathcal{V})^\tau$ as follows: $\alpha \succ \beta$ if either (i) $\alpha, \beta \in \mathcal{L}$ and $\alpha > \beta$, (ii) $\alpha \in \mathcal{L}$ and $\beta \in \mathrm{T}(\mathcal{F},\mathcal{V})^\tau$, or (iii) $\alpha, \beta \in \mathrm{T}(\mathcal{F},\mathcal{V})^\tau$ and $\alpha \xrightarrow{+}_{\mathcal{R}^\tau_{nl}} \beta$.*
3. *Let $\delta : \mathcal{R}^\tau_l \to \mathcal{L}$. The labelling function lab_δ from the rewrite steps of $\mathcal{R}^\tau$ to $\mathcal{L} \cup \mathrm{T}(\mathcal{F} \cup \mathcal{V})^\tau$ is given like this:*

$$lab_\delta(s \to_{l\to r} t) = \begin{cases} \delta(l \to r) & \text{if } l \to r \in \mathcal{R}^\tau_l \\ s & \text{if } l \to r \in \mathcal{R}^\tau_{nl} \end{cases}$$

The labelling given like $lab_\delta(s \to t) = s$ is called *source-labelling* [12]. Thus, our labelling is a combination of the rule-labelling and the source-labelling[1].

In the rest of this section, we assume that τ is a sort assignment consistent with $\mathcal{R}$ and that $\mathcal{R}^\tau$ is strongly quasi-linear. Furthermore, we suppose a set $\mathcal{L}$ of labels with a well-founded partial order $>$ and $\delta : \mathcal{R}^\tau_l \to \mathcal{L}$ are fixed.

The next lemma is an immediate corollary of well-foundedness of the partial order $>$ on $\mathcal{L}$ and the termination of $\mathcal{R}^\tau_{nl}$.

Lemma 3.3. *The relation $\succ$ on $\mathcal{L} \cup \mathrm{T}(\mathcal{F},\mathcal{V})^\tau$ is a well-founded partial order.*

[1] A similar idea has been adapted in Theorem 5 of [12].

It is trivial to show that disjoint peaks are decreasing. The proof that variable peaks are decreasing is straightforward but interesting, as it reveals why our choice of the labelling function matters (and other variations do not work).

Lemma 3.4. *Any disjoint peak is decreasing w.r.t.* lab_δ.

Lemma 3.5. *Any variable peak is decreasing w.r.t.* lab_δ.

Thus, it remains to show that overlap peaks are decreasing, but this does not hold in general. We reduce decreasingness of overlap peaks to that of critical peaks, where decreasingness of critical peaks is guaranteed by a sufficient criterion, which we introduce below.

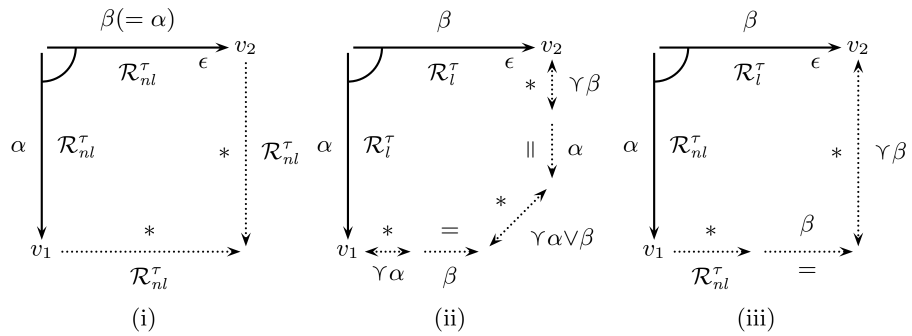

Fig. 2. Hierarchically decreasing critical peaks

Definition 3.6 (hierarchical decreasingness). *Any critical peak* $v_1 \leftarrow_{l_1 \to r_1} \circ \to_{l_2 \to r_2} v_2$ *is said to be* hierarchically decreasing w.r.t. δ and $>$ *if either one of the following conditions (i)–(iii) holds (Figure 2):*

(i) $l_1 \to r_1, l_2 \to r_2 \in \mathcal{R}^\tau_{nl}$ *and* $v_1 \xrightarrow{*} \circ \xleftarrow{*} v_2$ *(and hence* $v_1 \xrightarrow{*}_{\mathcal{R}^\tau_{nl}} \circ \xleftarrow{*}_{\mathcal{R}^\tau_{nl}} v_2$*).*

(ii) $l_1 \to r_1, l_2 \to r_2 \in \mathcal{R}^\tau_{l}$ *and* $v_1 \overset{*}{\leftrightarrow}_{\curlyvee\alpha} \circ \xrightarrow{=}_{\beta} \circ \overset{*}{\leftrightarrow}_{\curlyvee\alpha\vee\beta} \circ \xleftarrow{=}_{\alpha} \circ \overset{*}{\leftrightarrow}_{\curlyvee\beta} v_2$ *and*

(iii) $l_1 \to r_1 \in \mathcal{R}^\tau_{nl}$, $l_2 \to r_2 \in \mathcal{R}^\tau_{l}$ *and* $v_1 \xrightarrow{*}_{\mathcal{R}^\tau_{nl}} \circ \xrightarrow{=}_{\beta} \circ \overset{*}{\leftrightarrow}_{\curlyvee\beta} v_2$,

where $\alpha = \delta(l_1 \to r_1)$ *and* $\beta = \delta(l_2 \to r_2)$. *A many-sorted TRS* $\mathcal{R}^\tau$ *is said to be hierarchically decreasing (w.r.t.* δ *and* $>$*) if so are all critical peaks of* $\mathcal{R}^\tau$.

Note that the remaining case, i.e. the case of $l_1 \to r_1 \in \mathcal{R}^\tau_l$ and $l_2 \to r_2 \in \mathcal{R}^\tau_{nl}$ needs not be considered (see a remark below Example 2.4). It may look the conditions (i) and (iii) can be obtained by reducing the decreasingness by using the fact that any label of rewrite steps of $\mathcal{R}^\tau_{nl}$ is smaller than any label of rewrite steps of $\mathcal{R}^\tau_l$, but this is not true; in fact, these conditions are weaker than what are possible according to the definition of decreasingness.

The following properties are used to reduce the decreasingness of overlap peaks to that of critical peaks. For a rewrite step $\gamma : s \to_{l \to r} t$, a context C and a substitution θ, we put $C[\gamma\theta] : C[s\theta] \to_{l \to r} C[t\theta]$ ([19]).

Lemma 3.7. *Let γ, γ' be rewrite steps of $\mathcal{R}^\tau$, C a context, θ a substitution and $\propto \in \{=, \prec\}$. If $lab_\delta(\gamma) \propto lab_\delta(\gamma')$ then $lab_\delta(C[\gamma\theta]) \propto lab_\delta(C[\gamma'\theta])$.*

Lemma 3.8. *If $\mathcal{R}^\tau$ is hierarchically decreasing w.r.t. δ, then any overlap peak is decreasing w.r.t. lab_δ.*

Proof. It follows from the definition of hierarchical decreasingness that any critical peak is decreasing. Then the claim follows using Lemma 3.7. ☐

Now we arrive at the main theorem of this section.

Theorem 3.9 (confluence of strongly quasi-linear TRSs). *If $\mathcal{R}^\tau$ is strongly quasi-linear and hierarchically decreasing, then $\mathcal{R}$ is confluent.*

Proof. Every peak is decreasing w.r.t. lab_δ by Lemmas 3.4, 3.5 and 3.8. Thus, the claim follows from Propositions 2.1 and 2.2. ☐

Example 3.10. Let

$$\mathcal{R} = \left\{ \begin{array}{llll} (r1)\ \mathsf{f}(x, \mathsf{h}(x)) & \to \mathsf{f}(\mathsf{h}(x), \mathsf{h}(x)) & (r2)\ \mathsf{f}(x, \mathsf{k}(y, z)) & \to \mathsf{f}(\mathsf{h}(y), \mathsf{h}(y)) \\ (r3)\ \mathsf{h}(x) & \to \mathsf{k}(x, x) & (r4)\ \mathsf{k}(\mathsf{a}, \mathsf{a}) & \to \mathsf{h}(\mathsf{b}) \\ (r5)\ \mathsf{a} & \to \mathsf{b} & & \end{array} \right\}$$

We consider $\mathcal{S} = \mathcal{L} = \mathbb{N}$ and the standard relation $>$ on $\mathbb{N}$. Take a sort assignment $\tau = \{\mathsf{f} : 0 \times 0 \to 1, \mathsf{k} : 0 \times 0 \to 0, \mathsf{h} : 0 \to 0, \mathsf{a} : 0, \mathsf{b} : 0\}$ consistent with $\mathcal{R}$. Then $\mathcal{S}_{nl}(\mathcal{R}^\tau) = \{0\}$ and $\mathcal{R}^\tau_{nl} = \{(r3), (r4), (r5)\}$ is terminating. Thus $\mathcal{R}^\tau$ is strongly quasi-linear. Take $\delta = \{(r1) \mapsto 0, (r2) \mapsto 0\} : \mathcal{R}^\tau_l \to \mathbb{N}$. We have

$$\mathrm{CP}(\mathcal{R}) = \left\{ \begin{array}{ll} (cp1) & \mathsf{f}(x, \mathsf{k}(x, x)) \leftarrow_{(r3)} \mathsf{f}(x, \mathsf{h}(x)) \to_{(r1)} \mathsf{f}(\mathsf{h}(x), \mathsf{h}(x)) \\ (cp2) & \mathsf{f}(x, \mathsf{h}(\mathsf{b})) \leftarrow_{(r4)} \mathsf{f}(x, \mathsf{k}(\mathsf{a}, \mathsf{a})) \to_{(r2)} \mathsf{f}(\mathsf{h}(\mathsf{a}), \mathsf{h}(\mathsf{a})) \\ (cp3) & \mathsf{k}(\mathsf{b}, \mathsf{a}) \leftarrow_{(r5)} \mathsf{k}(\mathsf{a}, \mathsf{a}) \to_{(r4)} \mathsf{h}(\mathsf{b}) \\ (cp4) & \mathsf{k}(\mathsf{a}, \mathsf{b}) \leftarrow_{(r5)} \mathsf{k}(\mathsf{a}, \mathsf{a}) \to_{(r4)} \mathsf{h}(\mathsf{b}) \end{array} \right\}.$$

We now check that every critical peak is hierarchically decreasing.

- (cp1) We have $(r3) \in \mathcal{R}^\tau_{nl}$, $(r1) \in \mathcal{R}^\tau_l$ and $\delta((r1)) = 0$. Thus $\mathsf{f}(x, \mathsf{k}(x, x)) \leftarrow_{\mathcal{R}^\tau_{nl}} \circ \to_0 \mathsf{f}(\mathsf{h}(x), \mathsf{h}(x))$. Since $\mathsf{f}(x, \mathsf{k}(x, x)) \to_0 \mathsf{f}(\mathsf{h}(x), \mathsf{h}(x))$, the condition (iii) of hierarchical decreasingness holds.
- (cp2) We have $(r4) \in \mathcal{R}^\tau_{nl}$ $(r2) \in \mathcal{R}^\tau_l$ and $\delta((r2)) = 0$. $\mathsf{f}(x, \mathsf{h}(\mathsf{b})) \leftarrow_{\mathcal{R}^\tau_{nl}} \circ \to_0 \mathsf{f}(\mathsf{h}(\mathsf{a}), \mathsf{h}(\mathsf{a}))$. Since $\mathsf{f}(x, \mathsf{h}(\mathsf{b})) \to_{\mathcal{R}^\tau_{nl}} \mathsf{f}(x, \mathsf{k}(\mathsf{b}, \mathsf{b})) \to_0 \mathsf{f}(\mathsf{h}(\mathsf{b}), \mathsf{h}(\mathsf{b})) \leftarrow_{\mathcal{R}^\tau_{nl}} \mathsf{f}(\mathsf{h}(\mathsf{a}), \mathsf{h}(\mathsf{b})) \leftarrow_{\mathcal{R}^\tau_{nl}} \mathsf{f}(\mathsf{h}(\mathsf{a}), \mathsf{h}(\mathsf{a}))$, the condition (iii) of hierarchical decreasingness holds.
- (cp3), (cp4) We have $(r4), (r5) \in \mathcal{R}^\tau_{nl}$. It is easy to check the condition (i) of hierarchical decreasingness holds.

Thus, every critical peak is hierarchically decreasing. Hence, by Theorem 3.9, it follows that $\mathcal{R}$ is confluent.

4 Confluence of Quasi-Linear TRSs

In this section, we give a confluence criterion based on the decreasing diagrams and quasi-linearity, a notion for many-sorted TRSs obtained by replacing "termination of $\mathcal{R}^\tau_{nl}$" of strong quasi-linearity by "innermost normalization of $\mathcal{R}^\tau_{nl}$."

Definition 4.1 (quasi-linear). *A many-sorted TRS $\mathcal{R}^\tau$ is* quasi-linear *if the many-sorted TRS $\mathcal{R}^\tau_{nl}$ is innermost normalizing.*

Clearly, strongly quasi-linear (many-sorted) TRSs are quasi-linear but not vice versa.

To deal with non-linear TRSs, we here introduce a many-sorted linear TRS $\mathcal{R}^\tau_{nf}$, which is obtained by *instantiating* non-linear variables by ground normal forms. We will give a translation from a *quasi-linear* TRS $\mathcal{R}^\tau$ to a many-sorted TRS $\mathcal{R}^\tau_{nf}$ with *infinite* number of *linear* rewrite rules. Then we show that confluence of $\mathcal{R}^\tau_{nf}$ implies that of $\mathcal{R}^\tau$.

We will distinguish object terms to be rewritten and rewrite rules. To deal with confluence, variables in object terms can always be regarded as constants. Thus, consider constants c_x corresponding to each variable x, and let $\mathcal{C}_\mathcal{V} = \{c_x \mid x \in \mathcal{V}\}$. Now, we consider the set $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})$ as the set of object terms to be rewritten. Suppose $t \in \mathrm{T}(\mathcal{F}, \mathcal{V})$ and $\mathcal{V}(t) = \{x_1, \ldots, x_n\}$. Let t^c be the term in $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})$ obtained by replacing each x_i with c_{x_i} $(1 \leq i \leq n)$. Then $s \to_\mathcal{R} t$ iff $s^c \to_\mathcal{R} t^c$. Hence $\mathcal{R}$ is confluent on $\mathrm{T}(\mathcal{F}, \mathcal{V})$ iff $\mathcal{R}$ is confluent on $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})$. Similarly, by extending sort assignment τ by $\tau(c_x) = \tau(x)$, it follows that $\mathcal{R}^\tau$ is confluent on $\mathrm{T}(\mathcal{F}, \mathcal{V})^\tau$ iff $\mathcal{R}^\tau$ is confluent on $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$. Henceforth, let $\mathrm{NF}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$ be the set of normal forms from $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$ (w.r.t. $\to_{\mathcal{R}^\tau}$).

Definition 4.2 (linearization of quasi-linear TRSs). *Let $\mathcal{R}^\tau$ be a quasi-linear TRS. For $U^\tau \subseteq \mathcal{R}^\tau$, we define a many-sorted TRS U^τ_{nf} by*

$$U^\tau_{nf} = \bigcup_{l \to r \in U^\tau} \{l\hat{\theta} \to r\hat{\theta} \mid \hat{\theta} : \mathcal{V}_{nl}(l \to r) \to \mathrm{NF}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau\}.$$

(U^τ_{nf} is written as $\mathcal{R}^\tau_{nf}$ if we take $U^\tau = \mathcal{R}^\tau$.) We write a rewrite rule of U^τ_{nf} as $l\hat{\theta} \to r\hat{\theta}$, for brevity, to denote $l \to r \in U^\tau$ and $\hat{\theta} : \mathcal{V}_{nl}(l \to r) \to \mathrm{NF}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$.

Example 4.3. Let $\mathcal{S} = \{0, 1, 2\}$ and

$$\mathcal{R} = \left\{ (r1)\ \mathsf{f}(x, x, y) \to \mathsf{f}(x, \mathsf{g}(x), y) \quad (r2)\ \mathsf{f}(x, y, z) \to \mathsf{h}(\mathsf{a}) \right\}.$$

Take a sort assignment $\tau = \{\mathsf{f} : 0 \times 0 \times 1 \to 2, \mathsf{g} : 0 \to 0, \mathsf{h} : 0 \to 2, \mathsf{a} : 0\}$ consistent with $\mathcal{R}$. Since $\mathcal{V}_{nl}(r1) = \{x\}$ and $\mathcal{V}_{nl}(r2) = \emptyset$, we obtain $\mathcal{R}^\tau_{nf} = \{\mathsf{f}(s, s, y) \to \mathsf{f}(s, \mathsf{g}(s), y) \mid s \in \mathrm{NF}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau, \tau(s) = 0\} \cup \{\mathsf{f}(x, y, z) \to \mathsf{h}(\mathsf{a})\}$.

It is clear that $\mathcal{R}^\tau_{nf}$ is a linear TRS, as all non-linear variables of rewrite rules are instantiated by ground terms. Since there are infinitely many instantiations of each rewrite rule, $\mathcal{R}^\tau_{nf}$ has infinitely many numbers of rewrite rules.

In the rest of this section, we assume that τ is a sort assignment consistent with $\mathcal{R}$ and that $\mathcal{R}^\tau$ is quasi-linear. We also abbreviate $\to_{\mathcal{R}^\tau}$ and $\to_{\mathcal{R}^\tau_{nf}}$ by $\to$ and $\underset{nf}{\to}$, respectively. The next lemma is used to show Lemma 4.5.

Lemma 4.4. *Suppose* $s \xrightarrow{*}{}^{im} t$ *and let* $U^\tau = \mathit{Rules}(s \xrightarrow{*}{}^{im} t)$*. Then* $s \xrightarrow{*}_{U^\tau_{nf}} t$*.*

Lemma 4.5. *Let* $l \to r \in \mathcal{R}^\tau$ *and* $U^\tau = \{l \to r\}_{nl}$*. Then,* $\to_{l \to r} \subseteq \xrightarrow{*}_{U^\tau_{nf}} \circ \to_{\{l\to r\}_{nf}} \circ \xleftarrow{*}_{U^\tau_{nf}}$ *on* $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$*.*

A corollary of the previous lemma is the following sufficient criterion of confluence of $\mathcal{R}^\tau$ in terms of $\mathcal{R}^\tau_{nf}$, on which our analysis will be based.

Lemma 4.6. *A quasi-linear TRS* $\mathcal{R}^\tau$ *is confluent on* $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$ *if so is its linearization* $\mathcal{R}^\tau_{nf}$*.*

Proof. The claim easily follows from $\underset{nf}{\to} \subseteq \to \subseteq \underset{nf}{\overset{*}{\leftrightarrow}}$, which holds by Lemma 4.5 and the definition of $\mathcal{R}^\tau_{nf}$. □

The next lemma is used to analyze overlap peaks of $\mathcal{R}^\tau_{nf}$ by those of $\mathcal{R}^\tau$.

Lemma 4.7. *Let* $v_1 \leftarrow_{l_1\hat{\theta}_1 \to r_1\hat{\theta}_1} \circ \to_{l_2\hat{\theta}_2 \to r_2\hat{\theta}_2} v_2$ *be a critical peak of* $\mathcal{R}^\tau_{nf}$*. Then there exist a critical peak* $u_1 \leftarrow_{l_1 \to r_1} \circ \to_{l_2 \to r_2} u_2$ *of* $\mathcal{R}^\tau$ *and a substitution* θ *such that* $u_i\theta = v_i$ *(*$i = 1, 2$*).*

We note the converse of Lemma 4.7 does not hold in general.

Let $\mathcal{L}$ stand for the set of labels and $\succ$ be a well-founded partial order on $\mathcal{L}$.

Definition 4.8 (labelling on $\mathcal{R}^\tau_{nf}$). *Let* $lab : \mathcal{R}^\tau \to \mathcal{L}$*. We extend*[2] *lab to a function* $\mathcal{R}^\tau_{nf} \to \mathcal{L}$ *by* $lab(l\hat{\theta} \to r\hat{\theta}) = lab(l \to r)$*. Furthermore, we label the rewrite steps* $s \to_{l\hat{\theta} \to r\hat{\theta}} t$ *of* $\mathcal{R}^\tau_{nf}$ *by* $lab(l\hat{\theta} \to r\hat{\theta})$*.*

In the following, we assume some $lab : \mathcal{R}^\tau \to \mathcal{L}$ is fixed. Let ℓ (literally) stands for α or $\alpha \vee \beta$ ($\alpha, \beta \in \mathcal{L}$). For any $U^\tau \subseteq \mathcal{R}^\tau$, we write $U^\tau \prec \ell$ ($U^\tau_{nf} \prec \ell$) if $lab(l \to r) \prec \ell$ for all $l \to r \in U^\tau$ ($l \to r \in U^\tau_{nf}$, respectively). Note $U^\tau \prec \ell$ iff $U^\tau_{nf} \prec \ell$ for any $U^\tau \subseteq \mathcal{R}^\tau$. Let $\mathit{Rules}_{nl}(\gamma) = (\mathit{Rules}(\gamma))_{nl}$ for any conversion γ.

The next technical lemma, to be used in our key lemma (Lemma 4.11), is an immediate consequence of Lemma 4.5.

Lemma 4.9. *Let* $\gamma : s \overset{*}{\leftrightarrow}_{\curlyvee\ell} s' \overset{=}{\to}_\beta t$ *be a conversion on* $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$*. If* $\mathit{Rules}_{nl}(\gamma) \prec \ell$ *then* $s \underset{nf}{\overset{*}{\leftrightarrow}}_{\curlyvee\ell} \hat{s} \underset{nf}{\overset{=}{\to}}_\beta \hat{t} \underset{nf}{\overset{*}{\leftarrow}}_{\curlyvee\ell} t$ *on* $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$*. Furthermore,* $s' = t$ *implies* $\hat{s} = \hat{t}$*.*

Definition 4.10 (linearized-decreasingness). *Any critical peak* $v_1 \leftarrow_\alpha \circ \to_\beta v_2$ *of* $\mathcal{R}^\tau$ *is said to be* linearized-decreasing *w.r.t.* $lab : \mathcal{R}^\tau \to \mathcal{L}$ *and* $\succ$ *if there exists a conversion*

$$v_1 \overset{*}{\leftrightarrow}_{\curlyvee\alpha} \circ \overset{=}{\to}_\beta u_1 \overset{*}{\leftrightarrow}_{\curlyvee\alpha\vee\beta} u_2 \overset{=}{\leftarrow}_\alpha \circ \overset{*}{\leftrightarrow}_{\curlyvee\beta} v_2$$

on $\mathrm{T}(\mathcal{F}, \mathcal{V})^\tau$ *such that the following conditions (i)–(iii) are satisfied (Figure 3):*

[2] Thus, strictly speaking, $l\hat{\theta} \to r\hat{\theta}$ should be considered as $\langle l \to r, \hat{\theta}\rangle$, to distinguish common instances of different rewrite rules.

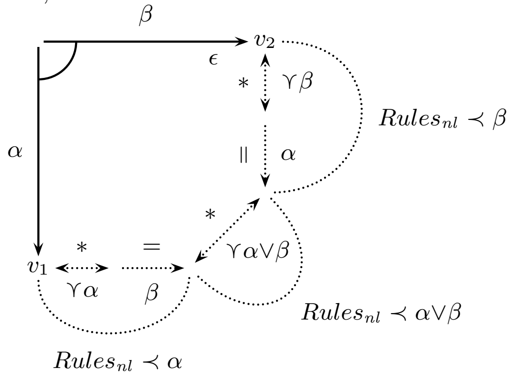

Fig. 3. linearized-decreasing critical peak

(i) $Rules_{nl}(v_1 \overset{*}{\leftrightarrow}_{\curlyvee\alpha} \circ \overset{=}{\rightarrow}_{\beta} u_1)) \prec \alpha$,
(ii) $Rules_{nl}(u_1 \overset{*}{\leftrightarrow}_{\curlyvee\alpha\vee\beta} u_2) \prec \alpha \vee \beta$, *and*
(iii) $Rules_{nl}(u_2 \overset{=}{\leftarrow}_{\alpha} \circ \overset{*}{\leftrightarrow}_{\curlyvee\beta} v_2) \prec \beta$.

A many-sorted TRS $\mathcal{R}^\tau$ is said to be linearized-decreasing (w.r.t. lab and $\succ$) if so are all critical peaks of $\mathcal{R}^\tau$.

Lemma 4.11. *Let $\mathcal{R}^\tau$ be a quasi-linear and linearized-decreasing TRS. Let $w_1 \leftarrow_\alpha \circ \rightarrow_\beta w_2$ be a critical peak of $\mathcal{R}^\tau_{nf}$ and $w_1\theta, w_2\theta \in \mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$. Then, $w_1\theta \underset{nf}{\overset{*}{\leftrightarrow}}_{\curlyvee\alpha} \circ \underset{nf}{\overset{=}{\rightarrow}}_{\beta} \circ \underset{nf}{\overset{*}{\leftrightarrow}}_{\curlyvee\alpha\vee\beta} \circ \underset{nf}{\overset{=}{\leftarrow}}_{\alpha} \circ \underset{nf}{\overset{*}{\leftrightarrow}}_{\curlyvee\beta} w_2\theta$ on $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$.*

Proof. Let $w_1 \leftarrow_{l_1\hat{\theta}_1 \rightarrow r_1\hat{\theta}_1} \circ \rightarrow_{l_2\hat{\theta}_2 \rightarrow r_2\hat{\theta}_2} w_2$. Then, by Lemma 4.7, there exists critical peak $v_1 \leftarrow_{l_1 \rightarrow r_1} \circ \rightarrow_{l_2 \rightarrow r_2} v_2$ such that $w_1 = v_1\theta'$ and $w_2 = v_2\theta'$ for some θ'. Then, by the definition of labelling of rewrite steps of $\mathcal{R}^\tau_{nf}$, we have $v_1 \leftarrow_\alpha \circ \rightarrow_\beta v_2$. Thus, by assumption, there exists a conversion $v_1 \overset{*}{\leftrightarrow}_{\curlyvee\alpha} \circ \overset{=}{\rightarrow}_{\beta} u_1 \overset{*}{\leftrightarrow}_{\curlyvee\alpha\vee\beta} u_2 \overset{=}{\leftarrow}_{\alpha} \circ \overset{*}{\leftrightarrow}_{\curlyvee\beta} v_2$ on $\mathrm{T}(\mathcal{F}, \mathcal{V})^\tau$ satisfying conditions (i)–(iii) of Definition 4.10. Now, apply the substitution $\theta \circ \theta'$ to this conversion to obtain

$$w_1\theta = v_1\theta'\theta \overset{*}{\leftrightarrow}_{\curlyvee\alpha} \circ \overset{=}{\rightarrow}_{\beta} u_1\theta'\theta \overset{*}{\leftrightarrow}_{\curlyvee\alpha\vee\beta} u_2\theta'\theta \overset{=}{\leftarrow}_{\alpha} \circ \overset{*}{\leftrightarrow}_{\curlyvee\beta} v_2\theta'\theta = w_2\theta.$$

Here, w.l.o.g. one can extend θ with $x \mapsto c_x$ so that this conversion is on $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$. Furthermore, conditions (i)–(iii) of Definition 4.10 imply

$$
\begin{aligned}
&(i') \;\; Rules_{nl}(v_1\theta'\theta \overset{*}{\leftrightarrow}_{\curlyvee\alpha} \circ \overset{=}{\rightarrow}_{\beta} u_1\theta'\theta) \prec \alpha,\\
&(ii') \;\; Rules_{nl}(u_1\theta'\theta \overset{*}{\leftrightarrow}_{\curlyvee\alpha\vee\beta} u_2\theta'\theta) \prec \alpha \vee \beta, \text{ and}\\
&(iii') \;\; Rules_{nl}(u_2\theta'\theta \overset{=}{\leftarrow}_{\alpha} \circ \overset{*}{\leftrightarrow}_{\curlyvee\beta} v_2\theta'\theta) \prec \beta.
\end{aligned}
$$

Then, by Lemma 4.9, $v_1\theta'\theta \underset{nf}{\overset{*}{\leftrightarrow}}_{\curlyvee\alpha} \circ \underset{nf}{\overset{=}{\rightarrow}}_{\beta} \circ \underset{nf}{\overset{*}{\leftarrow}}_{\curlyvee\alpha} u_1\theta'\theta \underset{nf}{\overset{*}{\leftrightarrow}}_{\curlyvee\alpha\vee\beta} u_2\theta'\theta \underset{nf}{\overset{*}{\rightarrow}}_{\curlyvee\beta} \circ \underset{nf}{\overset{=}{\leftarrow}}_{\alpha} \circ \underset{nf}{\overset{*}{\leftrightarrow}}_{\curlyvee\beta} v_2\theta'\theta$ on $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$. As $\underset{nf}{\leftarrow}_{\curlyvee\alpha}, \underset{nf}{\rightarrow}_{\curlyvee\beta} \subseteq \underset{nf}{\leftrightarrow}_{\curlyvee\alpha\vee\beta}$, the claim follows. □

Lemma 4.12. *Let $\mathcal{R}^\tau$ be a quasi-linear and linearized-decreasing TRS. Then $\underset{nf}{\rightarrow}$ on $\mathrm{T}(\mathcal{F} \cup \mathcal{C}_\mathcal{V})^\tau$ is locally decreasing.*

Proof. The claim follows easily for disjoint peaks. The case for variable peaks follows also easily, as $\mathcal{R}^\tau_{nf}$ is linear. The case for trivial overlap peaks are obvious. The case for non-trivial overlap peaks follows from Lemma 4.11. $\square$

Theorem 4.13 (confluence of quasi-linear TRSs). *If $\mathcal{R}^\tau$ is quasi-linear and linearized-decreasing then $\mathcal{R}$ is confluent.*

Proof. Use Lemmas 4.6 and 4.12 and Propositions 2.1 and 2.2. $\square$

Example 4.14. Let

$$\mathcal{R} = \left\{ \begin{array}{llll} (r1)\ \mathsf{f}(x,y) & \to \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) & (r2)\ \mathsf{f}(\mathsf{g}(x),x) & \to \mathsf{f}(x,\mathsf{g}(x)) \\ (r3)\ \mathsf{g}(x) & \to \mathsf{h}(x) & (r4)\ \mathsf{h}(\mathsf{g}(x)) & \to \mathsf{g}(\mathsf{g}(x)) \end{array} \right\}$$

We consider $\mathcal{S} = \mathcal{L} = \mathbb{N}$ and the standard relation $>$ on $\mathbb{N}$. Take a sort assignment $\tau = \{\mathsf{f} : 0 \times 0 \to 1, \mathsf{g} : 0 \to 0, \mathsf{h} : 0 \to 0\}$ consistent with $\mathcal{R}$. We have $\mathcal{S}_{nl}(\mathcal{R}^\tau) = \{0\}$ and $\mathcal{R}^\tau_{nl} = \{(r3), (r4)\}$ is innermost normalizing. Thus $\mathcal{R}^\tau$ is quasi-linear. We have

$$\mathrm{CP}(\mathcal{R}) = \left\{ \begin{array}{ll} (cp1) & \mathsf{f}(\mathsf{h}(x),x) \leftarrow_{(r3)} \mathsf{f}(\mathsf{g}(x),x) \to_{(r2)} \mathsf{f}(x,\mathsf{g}(x)) \\ (cp2) & \mathsf{h}(\mathsf{h}(x)) \leftarrow_{(r3)} \mathsf{h}(\mathsf{g}(x)) \to_{(r4)} \mathsf{g}(\mathsf{g}(x)) \\ (cp3) & \mathsf{f}(x,\mathsf{g}(x)) \leftarrow_{(r2)} \mathsf{f}(\mathsf{g}(x),x) \to_{(r1)} \mathsf{f}(\mathsf{g}(\mathsf{g}(x)),\mathsf{g}(\mathsf{g}(x))) \\ (cp3') & \mathsf{f}(\mathsf{g}(\mathsf{g}(x)),\mathsf{g}(\mathsf{g}(x))) \leftarrow_{(r1)} \mathsf{f}(\mathsf{g}(x),x) \to_{(r2)} \mathsf{f}(x,\mathsf{g}(x)) \end{array} \right\}.$$

Take $lab = \{(r1) \mapsto 2, (r2) \mapsto 3, (r3) \mapsto 0, (r4) \mapsto 1\} : \mathcal{R} \to \mathbb{N}$.

- (cp1) We have $\gamma_1 : \mathsf{f}(\mathsf{h}(x),x) \to_{(r1)} \mathsf{f}(\mathsf{g}(\mathsf{h}(x)),\mathsf{g}(\mathsf{h}(x))) \leftarrow_{(r1)} \mathsf{f}(\mathsf{h}(x),\mathsf{g}(x)) = \gamma_2 : \mathsf{f}(\mathsf{h}(x),\mathsf{g}(x)) \leftarrow_{(r3)} \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) \leftarrow_{(r1)} \mathsf{f}(x,\mathsf{g}(x))$, and $\mathit{Rules}_{nl}(\gamma_1) = \{(r3),(r4)\} \prec 0 \vee 3$ and $\mathit{Rules}_{nl}(\gamma_2) = \{(r3),(r4)\} \prec 3$. Thus the critical peak is linearized-decreasing.
- (cp2) We have $\gamma : \mathsf{h}(\mathsf{h}(x)) \leftarrow_{(r3)} \mathsf{g}(\mathsf{h}(x)) \leftarrow_{(r3)} \mathsf{g}(\mathsf{g}(x))$, and $\mathit{Rules}_{nl}(\gamma) = \emptyset$. Thus the critical peak is linearized-decreasing.
- (cp3) We have $\gamma : \mathsf{f}(x,\mathsf{g}(x)) \to_{(r1)} \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) \to_{(r1)} \mathsf{f}(\mathsf{g}(\mathsf{g}(x)),\mathsf{g}(\mathsf{g}(x)))$ Since $\mathit{Rules}_{nl}(\gamma) = \{(r3),(r4)\} \prec 2$, the critical peak is linearized-decreasing. The case (cp3′) follows similarly.

Hence, by Theorem 4.13, it follows that $\mathcal{R}$ is confluent.

We now remark that Theorems 3.9 and 4.13 are incomparable. First, $\mathcal{R}$ in Example 4.14 is not strongly quasi-linear, as $\mathcal{R}^\tau_{nl}$ is not terminating. Thus, Theorem 3.9 is not subsumed by Theorem 4.13. In the next example, we show that Theorem 4.13 is not subsumed by Theorem 3.9.

Example 4.15. Let us consider $\mathcal{R}$ in Example 3.10. First note that $\{(r1)\}_{nl} = \{(r2)\}_{nl} = \{(r3)\}_{nl} = \{(r3),(r4),(r5)\}$. We consider conversions for the critical peak (cp2): $v_1 = \mathsf{f}(x,\mathsf{h}(\mathsf{b})) \leftarrow_{(r4)} \mathsf{f}(x,\mathsf{k}(\mathsf{a},\mathsf{a})) \to_{(r2)} \mathsf{f}(\mathsf{h}(\mathsf{a}),\mathsf{h}(\mathsf{a})) = v_2$. It is easy to see $x \overset{*}{\leftrightarrow} t$ implies $x = t$ for any t. From this, it follows that the conversion has the form $v_1 \overset{*}{\leftrightarrow} w_1 \to_{(r2)} w_2 \overset{*}{\leftrightarrow} v_2$. We now consider decreasing conversion $v_1 \overset{*}{\leftrightarrow}_{\curlyvee lab(r4)} \circ \overset{=}{\to}_{lab(r2)} u_1 \overset{*}{\leftrightarrow}_{\curlyvee lab(r4) \vee lab(r2)} u_2 \overset{=}{\leftarrow}_{lab(r4)} u_3 \overset{*}{\leftrightarrow}_{\curlyvee lab(r2)} v_2$ and distinguish cases by in which part of this conversion the rewrite step $w_1 \to_{(r2)} w_2$ is involved.

- Case $w_1 \to w_2$ is in $v_1 \overset{*}{\leftrightarrow}_{\curlyvee lab(r4)} \circ \overset{=}{\to}_{lab(r2)} u_1$. Then by $(r4) \in \{(r2)\}_{nl}$, one requires $lab(r4) \prec lab(r4)$, which is impossible.
- Case $w_1 \to w_2$ is in $u_1 \overset{*}{\leftrightarrow}_{\curlyvee lab(r4) \vee lab(r2)} u_2$. Then one needs $lab(r2) \prec lab(r4) \vee lab(r2)$ and $lab(r4) \prec lab(r4) \vee lab(r2)$. This is again impossible.
- Case $w_1 \to w_2$ is in $u_2 \overset{=}{\leftarrow}_{lab(r4)} u_3$. This is impossible because of the direction of the rewrite steps does not coincide.
- Case $w_1 \to w_2$ is in $u_3 \overset{*}{\leftrightarrow}_{\curlyvee lab(r2)} v_2$. Then one needs $lab(r2) \prec lab(r2)$, which is impossible.

Thus, the critical peak (cp2) is not linearized-decreasing.

Relations between Theorem 4.13 and Theorem 3.9 for some particular classes of TRSs follow. For non-overlapping TRSs, Theorem 4.13 is strictly subsumed by Theorem 3.9. For linear TRSs, Theorem 4.13 and Theorem 3.9 (and the original rule-labelling) are equivalent.

Finally, we note that one can generally include linear rules to $\mathcal{R}_{nl}^{\tau}$ in Theorem 3.9, and then Knuth-Bendix's criterion is obtained from Theorem 3.9. This is, however, not surprising as it is known that Knuth-Bendix's criterion can be given by decreasing diagrams with the source-labelling (Example 12 of [12]).

5 Implementation, Experiments and Related Work

The confluence criteria of the paper have been implemented in the confluence prover ACP [3]. We straightforwardly adapt techniques for automating decreasing diagrams based on rule-labelling [1, 8]. We use SML/NJ [13] for the implementation language and the constraint solver Yices [5] to check the satisfiability of constraints encoding existence of a labelling function satisfying our criteria.

Some heuristics and approximation employed in our implementation follow. To construct many-sorted TRS $\mathcal{R}^{\tau}$ from an unsorted TRS $\mathcal{R}$, it suffices to compute sort assignment τ consistent with $\mathcal{R}$. In practice, its enough to choose such a sort assignment that maximally distinct sorts, in order to maximize the applicability of the criteria. This can be done by first assigning fresh sorts for each sort declarations of function symbols and for variables, and then solving the constraint on these sorts that arises from the requirement that lhs and rhs of each rewrite rule are well-sorted terms having the same sort. To check the quasi-linearity of TRSs, one has to check innermost normalization of TRSs. To the best of our knowledge, no works concentrated on proving innermost normalization are known; thus, the check is approximated by checking innermost termination. To check decreasing diagram criteria, one has to find, for each critical peak $v_1 \leftarrow u \to v_2$, some conversions $v_1 \overset{*}{\leftrightarrow} v_2$ that are used as the candidates for $v_1 \overset{*}{\leftrightarrow}_{\curlyvee\alpha} \circ \overset{=}{\to}_{\beta} u_1 \overset{*}{\leftrightarrow}_{\curlyvee\alpha\vee\beta} u_2 \overset{=}{\leftarrow}_{\alpha} \circ \overset{*}{\leftrightarrow}_{\curlyvee\beta} v_2$. For this, our implementation uses sets of conversions $v_1 \overset{\leq 4}{\to}_{\mathcal{R}^{\leftrightarrow}} \circ \overset{\leq 4}{\leftarrow}_{\mathcal{R}^{\leftrightarrow}} v_2$ as the sets of candidates, where $\mathcal{R}^{\leftrightarrow} = \mathcal{R} \cup \{r \to l \mid l \to r \in \mathcal{R}, r \notin \mathcal{V}, \mathcal{V}(l) \subseteq \mathcal{V}(r)\}$ [8] and $s \overset{\leq 4}{\to} t$ means $s \overset{*}{\to} t$ in less than four rewrite steps. Then applicability of the criterion for all possible choice of u_1, u_2 in these sequences is encoded in a constraint [1].

Table 1. Experiments with the state-of-art confluence provers

	ACP	CSI	Saigawa	Thm. 3.9	Thm. 4.13	Thm. 3.9&4.13
Example 3.10	×	×	×	✓	×	✓
Example 4.14	×	×	×	×	✓	✓
9 examples from [14]	8	1	1	9	9	9
11 new examples	0	0	0	9	10	11

In [14], a critical pair criterion for quasi-left-linear TRSs have been given. The main differences between quasi-left-linear TRSs and quasi-linear TRSs are that (1) the former considers only non-linear variables on lhs of the rewrite rules while the latter considers non-linear variables on lhs or rhs of the rewrite rules, and (2) in the the former, $\mathcal{R}^{\tau}_{nf}$ is obtained by instantiating all variables of non-left-linear sorts, while in the latter $\mathcal{R}^{\tau}_{nf}$ is obtained by instantiating non-linear variables. We have adapted decreasing diagrams with rule-labelling for proving confluence of $\mathcal{R}^{\tau}_{nf}$ but in [14] critical pair criteria for left-linear TRSs (e.g. [9, 15]) are applied.

We now report on experiments for the collection of 9 examples from [14], and 11 new examples constructed in the course of experiments, including Examples 3.10 and 4.14. These are all non-left-linear and non-terminating TRSs. Tests are performed on a PC with one 2.50GHz CPU and 4G memory; the timeout is set to 60 seconds. For comparison with the state-of-art confluence provers, ACP (ver. 0.41) [3], CSI (ver. 0.4.1) [18] and Saigawa (ver. 1.5) [7] are used. The summary of experiments is shown in Table 1. For examples, ✓ denotes success and × denotes failure. Examples 3.10 and 4.14 are solved by none of the state-of-art confluence provers. For the collections, the number of successes is shown. ACP implements the technique given in [14], and thus can solve all examples but the last one which has been left open for automated confluence proving in [14]. Both of our new criteria prove this last example from [14]. All provers fail at solving all our new examples. Hence, in particular, the technique of [14] is not effective for all of our new examples. The difference of Thm. 3.9 and Thm. 4.13 appears on only few examples. In particular, Example 3.10 is only solved by the criterion of Thm. 3.9, and Example 4.14 is only solved by the criterion of Thm. 4.13.

Next we discuss other related work. In [1, 12, 19], the rule-labelling is extended to (non-linear) left-linear TRSs, where the one in latest [19] subsumes those in the others. This technique essentially depends on the path information to the duplicating variables in the rewrite rules to ensure the decreasingness of variable peaks. In [10], a criterion for proving confluence of non-left-linear TRSs using relative termination has been developed, whose correctness is proved based on the decreasing diagrams with source-labelling. This criterion essentially requires termination of $\mathcal{R}_1$ relative to $\mathcal{R}_2$ to show the confluence of $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$. Several other confluence criteria applicable for non-left-linear TRSs have been developed in [6, 17]; these criteria require some restrictions on the form of rewrite rules. All of these criteria are incomparable with the techniques developed in the present

Table 2. Experiments of confluence criteria for non-left-linear non-terminating TRSs

Criterion (or tool)	42 Cops	10 Cops
Thm. 3.9	13	1
Thm. 4.13	12	1
Criterion for quasi-left-linear TRSs [14] (in ACP)	9	0
Criterion for weight-decreasing TRSs [6] (in ACP)	13	0
Criterion for simple-right-linear TRSs [17] (in ACP)	1	0
Saigawa (including [10])	12	0

paper is witnessed in Table 1, as (the automatizable parts of) these techniques have been involved in some of the confluence provers ACP, CSI and Saigawa.

Next we compare strength of confluence criteria for non-terminating non-left-linear TRSs experimentally. For this, we use two collections of non-terminating non-left-linear problems from Cops (Confluence Problems) database: (i) the collection of 42 problems from CoCo 2013 that are not solved as '`non_confluent`' by any tool, and (ii) the collection of 10 problems from CoCo 2013 that are not solved by any tool. The criterion of [10] is approximated by Saigawa (Saigawa does not facilitate to choose a single technique employed). For other criteria, ACP is adapted to single out each criterion. In Table 2, the numbers of successes for each criterion are shown. We observe that the problems in CoCo 2013 do not differentiate strength of most of techniques very much. We note the current implementation of the technique of [17] in ACP is not very elaborated. The problem that is not solved in any provers in CoCo 2013 but solved by our new criteria is the last example from [14] mentioned before.

The collection of new examples and details of the experiments are available on the webpage `http://www.nue.riec.tohoku.ac.jp/tools/acp/experiments/rtatlca14/all.html`.

In this paper, sort constraint is used to limit instantiations of non-linear variables of rewrite rules. Imposing such limitation more abstractly leads to the framework of membership conditional rewriting systems and a confluence criterion for such systems [16].

6 Conclusion

We have presented two criteria for confluence of TRSs $\mathcal{R}$ based on decreasing diagrams with rule-labelling and persistency: (1) $\mathcal{R}^\tau$ is strongly quasi-linear and hierarchically decreasing, and (2) $\mathcal{R}^\tau$ is quasi-linear and linearized-decreasing. We have also shown that these criteria are incomparable. These criteria are particularly useful for proving non-linear TRSs confluent, including non-terminating non-left-linear TRSs for which only few confluence criteria have been known. Our criteria have been implemented in the confluence prover ACP. We have shown that our criteria are successfully used in confluence provers for proving confluence of TRSs for which none of the state-of-art confluence provers succeed.

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