Natural Inductive Theorems for Higher-Order Rewriting

Takahito Aoto[1], Toshiyuki Yamada[2], and Yuki Chiba[3]

1 **RIEC, Tohoku University**
2-1-1 Katahira, Aoba-ku, Sendai, Miyagi, 980-8577, Japan
aoto@nue.riec.tohoku.ac.jp

2 **Graduate School of Engineering, Mie University**
1577 Kurimamachiya, Tsu, Mie, 514-8507, Japan
toshi@cs.info.mie-u.ac.jp

3 **School of Information Science, Japan Advanced Institute of Science and Technology**
1-1 Asahidai, Nomi, Ishikawa, 923-1292, Japan
chiba@jaist.ac.jp

Abstract

The notion of inductive theorems is well-established in first-order term rewriting. In higher-order term rewriting, in contrast, it is not straightforward to extend this notion because of extensionality (Meinke, 1992). When extending the term rewriting based program transformation of Chiba et al. (2005) to higher-order term rewriting, we need extensibility, a property stating that inductive theorems are preserved by adding new functions via macros. In this paper, we propose and study a new notion of inductive theorems for higher-order rewriting, *natural inductive theorems*. This allows to incorporate properties such as extensionality and extensibility, based on simply typed S-expression rewriting (Yamada, 2001).

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1 Introduction

Properties of programs are often proved by induction on data structures such as natural numbers or lists. In the case of first-order term rewriting, such properties are captured by the notion of *inductive theorems* (e.g. [5]): an equation $s \approx t$ is said to be an inductive theorem of a term rewriting system (TRS for short) $\mathcal{R}$ if all ground instances are equational consequences, i.e. $s\theta \leftrightarrow^*_{\mathcal{R}} t\theta$ holds for any ground substitution θ. Inductive theorems form the initial semantics of first-order equational theories. In the higher-order case, one often expects *extensionality*, meaning that expressions denoting the same function are equivalent. The proof system and semantics of higher-order equational theories as well as the initial semantics of such theories based on *extensional inductive theorems* have been studied in [16, 17, 18]. In the simply typed S-expression rewriting framework [1, 2, 3, 21], the notion of higher-order inductive theorems and *inductionless induction* [12, 14, 15, 19] for proving higher-order inductive theorems automatically have been studied in [4].

Several transformations for optimizing functional programs have been developed [6, 10, 11, 13, 20]. One such framework is *program transformation by templates*, proposed by Huet and Lang [13]. Chiba et al. [7, 8, 9] developed a framework of program transformation by templates based on first-order term rewriting. In this framework, the correctness of the transformation—the equivalence of input and output TRSs—is formalized based on inductive equality. One of the ingredients for ensuring the correctness of this program transformation is *extensibility* of inductive theorems, meaning that inductive theorems are preserved when a new function by a macro (i.e. non-recursive function in terms of existing functions) is added.

In the case of higher-order term rewriting, in contrast, extensibility of extensional inductive theorems is not guaranteed. Consider the following simply typed S-expression rewriting system (STSRS for short):

$$\mathcal{R} = \left\{ \begin{array}{lcl} +\ \mathsf{0}\ y & \to & y \\ +\ (\mathsf{s}\ x)\ y & \to & \mathsf{s}\ (+\ x\ y) \\ \mathsf{zero}\ \mathsf{s} & \to & \mathsf{0} \end{array} \right\}.$$

Then $+\ x\ y \approx +\ y\ x$ is an extensional inductive theorem of $\mathcal{R}$, that is, for any ground substitution θ, $(+\ x\ y)\theta \stackrel{\mathrm{ext}}{\leftrightarrow}{}^{*}_{\mathcal{R}} (+\ y\ x)\theta$ holds. Here $\stackrel{\mathrm{ext}}{\leftrightarrow}{}^{*}_{\mathcal{R}}$ is an equivalence relation induced by $\mathcal{R}$ where extensionality is taken into account. However, if we add a new constant f and a rewrite rule $\mathsf{f}\ x \to \mathsf{0}$ to $\mathcal{R}$, then this does not hold anymore. For, we do not have $+\ (\mathsf{zero}\ \mathsf{f})\ \mathsf{0} \stackrel{\mathrm{ext}}{\leftrightarrow}{}^{*}_{\mathcal{R}} +\ \mathsf{0}\ (\mathsf{zero}\ \mathsf{f})$. Hence, the equation $+\ x\ y \approx +\ y\ x$ is not an inductive theorem of $\mathcal{R} \cup \{\mathsf{f}\ x \to \mathsf{0}\}$.

To see why extensibility is needed, consider the following program transformation. The *recursive* definition of rev, given by $\mathcal{R}_{in}$, is transformed into the *iterative* definition, given by $\mathcal{R}_{out}$. Both TRSs are first-order and given by:

$$\mathcal{R}_{in} = \left\{ \begin{array}{ll} \mathsf{rev}([\,]) & \to [\,] \\ \mathsf{rev}(x : xs) & \to \mathsf{app}(\mathsf{rev}(xs), x : [\,]) \\ \mathsf{app}([\,], ys) & \to ys \\ \mathsf{app}(x : xs, ys) & \to x : \mathsf{app}(xs, ys) \end{array} \right\} \Rightarrow \mathcal{R}_{out} = \left\{ \begin{array}{ll} \mathsf{rev}(xs) & \to \mathsf{rev1}(xs, [\,]) \\ \mathsf{rev1}([\,], ys) & \to ys \\ \mathsf{rev1}(x : xs, ys) & \to \mathsf{rev1}(xs, x : ys) \\ \mathsf{app}([\,], ys) & \to ys \\ \mathsf{app}(x : xs, ys) & \to x : \mathsf{app}(xs, ys) \end{array} \right\}.$$

The correctness of the transformation is guaranteed by the fact that the equations $\mathsf{app}(xs, [\,]) \approx xs$ and $\mathsf{app}(\mathsf{app}(xs, ys), zs) \approx \mathsf{app}(xs, \mathsf{app}(ys, zs))$ are inductively valid w.r.t. the input TRS $\mathcal{R}_{in}$. The transformation is carried out in three steps: $\mathcal{R}_{in} \stackrel{*}{\Rightarrow}_I \mathcal{R}_I \stackrel{*}{\Rightarrow}_A \mathcal{R}_A \stackrel{*}{\Rightarrow}_E \mathcal{R}_{out}$. In the first step, the definition of a new function $\mathsf{rev1}$ is introduced as $\mathsf{rev1}(xs, ys) \to \mathsf{app}(\mathsf{rev}(xs), ys)$. Note that the definition of $\mathsf{rev1}$ given here is defined in terms of the original rev and app functions and is different from the final form occurring in $\mathcal{R}_{out}$ which is defined recursively. In the second step, new rewrite rules which are inductively valid are added. For example, the rewrite rule $\mathsf{rev1}(x : xs, ys) \to \mathsf{app}(\mathsf{rev}(xs), x : ys)$ is added based on the inductive equivalence $\mathsf{rev1}(x : xs, ys) \leftrightarrow_{\mathcal{R}_I} \mathsf{app}(\mathsf{rev}(x : xs), ys) \leftrightarrow_{\mathcal{R}_I} \mathsf{app}(\mathsf{app}(\mathsf{rev}(xs), x : [\,]), ys) \approx \mathsf{app}(\mathsf{rev}(xs), \mathsf{app}(x : [\,], ys)) \leftrightarrow_{\mathcal{R}_I} \mathsf{app}(\mathsf{rev}(xs), x : \mathsf{app}([\,], ys)) \leftrightarrow_{\mathcal{R}_I} \mathsf{app}(\mathsf{rev}(xs), x : ys)$. Likewise, $\mathsf{rev}(xs) \to \mathsf{rev1}(xs, [\,])$ and $\mathsf{rev1}([\,], ys) \to ys$ are added. In the last step, auxiliary rewrite rules (typically original rules) are eliminated. By extensibility of first-order inductive theorems, the inductive theorems of $\mathcal{R}_{in}$ are still inductively valid in $\mathcal{R}_I$, and thus one can use inductive theorems safely in the second step $\mathcal{R}_I \stackrel{*}{\Rightarrow}_A \mathcal{R}_A$, after the introduction of $\mathsf{rev1}$ in the first step.

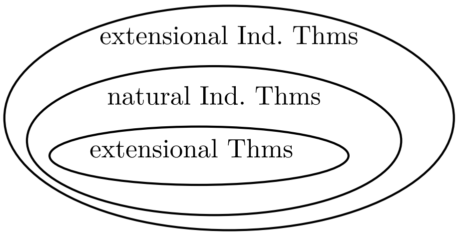

Figure 1 Inclusion relation on the three notions of theorems

The lack of extensibility for higher-order inductive theorems prevents us from extending the template based framework for program transformations of [7, 8, 9] to the higher-order setting. To overcome this difficulty, we introduce in this paper a new notion of inductive theorems—*natural inductive theorems*—for higher-order rewriting satisfying the following properties: (1) these inductive theorems are extensional and extensible, (2) extensional theorems are natural inductive theorems, and (3) natural inductive theorems are extensional inductive theorems (see Figure 1). Once the notion of natural inductive theorems is obtained, the higher-order extension of the framework is achieved in the following way. As in the first-order case, we first establish some natural inductive theorems of the input STSRS $\mathcal{R}_{in}$. Then a transformation $\mathcal{R}_{in} \stackrel{*}{\Rightarrow}_I \mathcal{R}_I \stackrel{*}{\Rightarrow}_A \mathcal{R}_A \stackrel{*}{\Rightarrow}_E \mathcal{R}_{out}$ is performed as before. By extensibility, natural inductive theorems are preserved in the transformation $\stackrel{*}{\Rightarrow}_I$. This, together with the property (2), allows to add new rules which are sound w.r.t. natural inductive validity. Hence the equivalence of $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ is obtained w.r.t. natural inductive validity. By property (3) this ensures the equivalence of input and output STSRSs w.r.t. extensional inductive validity.

The remainder of this paper is structured as follows. Having fixed the terminology and notations used in this paper (Section 2), we review a semantics of simply typed equational theories that captures extensionality (Section 3). In Section 4, we arrive at the restriction of simply typed algebras to give a notion of natural inductive theorems. Then we show that the set of natural inductive theorems covers that of extensional theorems and is covered by that of extensional inductive theorems. We then show extensibility of natural inductive theorems under certain conditions. In Section 5, we give a sufficient condition that partially allows to check whether an equation is a natural inductive theorem. Section 6 concludes.

2 Preliminaries

In this section, we briefly recall the terminology and notations of simply typed S-expression rewriting (simply typed term rewriting in [21]).

Let B be a set of *base types*. The set ST of simple types is defined inductively as: $B \subseteq \mathrm{ST}$; if $\tau_0, \ldots, \tau_n \in \mathrm{ST}$ then $\tau_1 \times \cdots \times \tau_n \to \tau_0 \in \mathrm{ST}$ ($n \geq 1$). Non-base types are called *function types*. A set $T \subseteq \mathrm{ST}$ of simple types is a *simple type structure* if (1) $B \subseteq T$ and (2) T is closed under subtypes, i.e. $\tau_1 \times \cdots \times \tau_n \to \tau_0 \in T$ implies $\tau_0, \ldots, \tau_n \in T$. For any simple type structure T, we put $T^{\mathrm{f}} = T \setminus B$. *Second-order* simple types are defined inductively as follows: (1) base types are second-order simple types, (2) if τ_0 is a second-order simple type and $\tau_1, \ldots, \tau_n$ are base types then $\tau_1 \times \cdots \times \tau_n \to \tau_0$ is a second-order simple type. Let Σ be a set of *constants* and V the set of *variables*. Each constant or variable a is equiped with a simple type (denoted by type(a)). We assume that there are countably infinite variables of type τ for each $\tau \in \mathrm{ST}$. For any $\tau \in \mathrm{ST}$ and $U \subseteq \Sigma \cup V$, we put $U^\tau = \{a \in U \mid \mathrm{type}(a) = \tau\}$. Let T be a simple type structure. We say $a \in \Sigma \cup V$ ($U \subseteq \Sigma \cup V$) is *over* T if type$(a) \in T$ ($U \subseteq \bigcup_{\tau \in T} U^\tau$, respectively). A simply typed constant is said to be second-order if its type is second-order. We assume that the set Σ of constants is partitioned into two categories[1]: the set Σ_d of *defined* constants and the set Σ_c of *constructor* constants. The set Σ is said to be *elementary* if any constructor constant $c \in \Sigma_\mathrm{c}$ is second-order.

Let Σ be a set of constants over a simple type structure T and X be a set of variables over T. The set $\mathrm{S}(\Sigma, X)^\tau$ of *simply typed S-expressions* of type $\tau \in T$ over Σ and X is defined as follows: (1) $\Sigma^\tau \cup X^\tau \subseteq \mathrm{S}(\Sigma, X)^\tau$, (2) if $t_0 \in \mathrm{S}(\Sigma, X)^{\tau_1 \times \cdots \times \tau_n \to \tau}$ and $t_i \in \mathrm{S}(\Sigma, X)^{\tau_i}$ for all $i \in \{1, \ldots, n\}$ then $(t_0\ t_1\ \cdots\ t_n) \in \mathrm{S}(\Sigma, X)^\tau$. The outermost parentheses of an S-expression can be omitted. The set of all simply typed S-expressions over Σ and X is denoted by $\mathrm{S}(\Sigma, X)$. We often refer to simply typed S-expressions as S-expressions, for brevity. The type of an S-expression t is denoted by type(t). For any set $U \subseteq \mathrm{S}(\Sigma, V)$, we put $U^\mathrm{b} = \{s \in U \mid \mathrm{type}(s) \in B\}$ and $U^\mathrm{f} = \{s \in U \mid \mathrm{type}(s) \notin B\}$. The set of variables in an S-expression t (of base type, of function type) is denoted by $\mathrm{V}(t)$ ($\mathrm{V}^\mathrm{b}(t)$, $\mathrm{V}^\mathrm{f}(t)$, respectively). An S-expression t is said to be *ground* if $\mathrm{V}(t) = \emptyset$. The set of ground S-expressions is denoted by $\mathrm{S}(\Sigma)$. An S-expression is *linear* if every variable occurs at most once in it. The *head symbol* of an S-expression is defined recursively as follows: head$(a) = a$ for $a \in \Sigma \cup V$; head$((t_0\ t_1 \cdots t_n))$ = head(t_0). The set Args(s) of arguments of an S-expression s is defined recursively as follows: Args$(a) = \emptyset$ for $a \in \Sigma \cup V$; Args$((t_0\ t_1 \cdots t_n))$ = Args$(t_0) \cup \{t_1, \ldots, t_n\}$. A *full expansion* $t\!\uparrow$ of an S-expression t is defined recursively as follows: (1) if type$(t) \in B$ then $t\!\uparrow = t$, (2) if type$(t) = \tau_1 \times \cdots \times \tau_n \to \tau_0$ then $t\!\uparrow = (t\ x_1 \cdots x_n)\!\uparrow$ where $x_1, \ldots, x_n$ are fresh variables of type $\tau_1, \ldots, \tau_n$, respectively.

A simply typed *context* over Σ and X is a simply typed S-expression over Σ and X that contains special symbols $\Box^\tau$, called the *holes*, prepared for each type $\tau \in T$. Let C be a context having a hole of type τ. The S-expression obtained by replacing the hole in C with an S-expression t of the same type is denoted by $C[t]$. A context of the form $\Box^\tau$ is said to be *empty*. We omit the type of a hole when it is not important. An S-expression s is a *subexpression* of an S-expression t (denoted by $s \trianglelefteq t$) if $C[s] = t$ for some context $C[\,]$.

A simply typed *substitution* over Σ is a mapping $\sigma : V \to \mathrm{S}(\Sigma, V)$ such that type(x) = type$(\sigma(x))$ for all $x \in V$ and $\mathrm{dom}(\sigma) = \{x \mid \sigma(x) \neq x\}$ is finite. The set dom(σ) is called the *domain* of σ. The *range* of σ is given by $\mathrm{ran}(\sigma) = \{\sigma(x) \mid x \in \mathrm{dom}(\sigma)\}$. For a substitution σ, we write $\sigma : U \to W$ if $\mathrm{dom}(\sigma) \subseteq U$ and $\mathrm{ran}(\sigma) \subseteq W$. As usual, we identify a substitution with its homomorphic extension.

An *instance* of an S-expression t is written as $t\sigma$. When we write $t\sigma$ for a substitution $\sigma : U \to W$, we assume that $\mathrm{V}(t) \cap U \subseteq \mathrm{dom}(\sigma)$. A substitution σ is ground if $\mathrm{ran}(\sigma) \subseteq \mathrm{S}(\Sigma)$. For a set $Y \subseteq X$ and a substitution σ over Σ and X, $\sigma\restriction_Y$ denotes a substitution given by $\sigma\restriction_Y(x) = \sigma(x)$ for $x \in Y$, $\sigma\restriction_Y(x) = x$ otherwise.

Let Σ be a set of constants over T. A *simply typed rewrite rule* $l \to r$ over Σ is a pair of simply typed S-expressions over Σ and $X = \bigcup_{\tau \in T} V^\tau$ which satisfies the following conditions: (1) type(l) = type(r), (2) head$(l) \in \Sigma$ and (3) $\mathrm{V}(r) \subseteq \mathrm{V}(l)$. A set $\mathcal{R}$ of rewrite rules over Σ is called a *simply typed S-expression rewriting system* (STSRS for short) over Σ. Let Y be a set of variables over T. For any $s, t \in \mathrm{S}(\Sigma, Y)$, we have $s \to_\mathcal{R} t$ if $s = C[l\sigma]$ and $t = C[r\sigma]$ for some rewrite rule $l \to r \in \mathcal{R}$, context $C[\,]$ over Σ and Y, and substitution $\sigma : X \to \mathrm{S}(\Sigma, Y)$.

[1] We do not assume in this paper that Σ_d coincides with the set of head symbols of left-hand sides of the rewrite rules, i.e. $\{\mathrm{head}(l) \mid l \to r \in \mathcal{R}\} = \Sigma_\mathrm{d}$.

The relation $\to_{\mathcal{R}}$ (over $\mathrm{S}(\Sigma, Y)$) is called the *rewrite relation* induced by an STSRS $\mathcal{R}$. An STSRS $\mathcal{R}$ is *left-linear* if l is linear for any $l \to r \in \mathcal{R}$. The symmetric closure and the reflexive transitive closure of a relation $\to$ is denoted by $\leftrightarrow$ and $\to^*$, respectively.

A *simply typed equation* over Σ and X is a pair $\langle l, r\rangle$ of simply typed S-expressions over Σ and X such that $\mathrm{type}(l) = \mathrm{type}(r)$. We write $l \approx r$ to denote that $\langle l, r\rangle$ is a simply typed equation. The set of simply typed equations over Σ and X is denoted by $\mathrm{Eqn}(\Sigma, X)$. For any $s \approx t \in \mathrm{Eqn}(\Sigma, X)$ a full expansion $s{\uparrow} \approx t{\uparrow}$ of $s \approx t$ is defined similarly to the full expansion of an S-expression by choosing the same variables in corresponding arguments in left-hand sides (lhss) and right-hand sides (rhss) of the equation. Any $E \subseteq \mathrm{Eqn}(\Sigma, X)$ where Σ is a set of constants over T and X is the set of variables over T is called a $\langle T, \Sigma\rangle$*-theory*. We sometime refer to an STSRS $\mathcal{R}$ over Σ as a $\langle T, \Sigma\rangle$-theory given by $\{l \approx r \mid l \to r \in \mathcal{R}\}$. A $\langle T, \Sigma\rangle$-theory is said to be *elementary* if Σ is elementary.

3 Extensional Semantics

In this section, we present a semantics for simply typed equational theories that captures extensionality and recall some basic results that will be used in the next section. Most of the material is incorporated from [17] into our framework.

▶ **Definition 3.1** (typed algebras). Let Σ be a set of simply typed constants over a simple type structure T. A T-typed Σ-algebra ($\langle T, \Sigma\rangle$-algebra for short) is a triple

$$\mathcal{A} = \langle (A^\tau)_{\tau \in T}, (ap^\tau)_{\tau \in T^{\mathrm{f}}}, (c^{\mathcal{A}})_{c \in \Sigma}\rangle$$

where $(A^\tau)_{\tau \in T}$ are mutually disjoint non-empty sets, $ap^\tau \in [A^\tau \times A^{\tau_1} \times \cdots \times A^{\tau_n} \to A^{\tau_0}]$ for each $\tau = \tau_1 \times \cdots \times \tau_n \to \tau_0 \in T^{\mathrm{f}}$, and $c^{\mathcal{A}} \in A^{\mathrm{type}(c)}$ for each $c \in \Sigma$. Here, for sets $A_0, \ldots, A_n$, $[A_1 \times \cdots \times A_n \to A_0]$ is the set of functions from $A_1 \times \cdots \times A_n$ to A_0. The set $\bigcup_{\tau \in T} A^\tau$ is called the *carrier set* of the algebra $\mathcal{A}$ and denoted by $|\mathcal{A}|$.

We now incorporate standard notions on the validity and equational consequences for our semantics. Let $\mathcal{A} = \langle (A^\tau)_{\tau \in T}, (ap^\tau)_{\tau \in T^{\mathrm{f}}}, (c^{\mathcal{A}})_{c \in \Sigma}\rangle$ be a $\langle T, \Sigma\rangle$-algebra and X the set of variables over T. A family of mappings $\rho = (\rho^\tau)_{\tau \in T}$ where $\rho^\tau \in [X^\tau \to A^\tau]$ is called an *environment* for $\mathcal{A}$. We abbreviate $\rho^\tau(x)$ as $\rho(x)$. For each S-expression $s \in \mathrm{S}(\Sigma, X)$ its *interpretation* $[\![s]\!]_\rho$ in $\mathcal{A}$ over the environment ρ is defined inductively like this: $[\![c]\!]_\rho = c^{\mathcal{A}}$ for each $c \in \Sigma$, $[\![x]\!]_\rho = \rho(x)$ for each $x \in X$, $[\![(s_0\ s_1 \cdots s_n)]\!]_\rho = ap^\tau([\![s_0]\!]_\rho, [\![s_1]\!]_\rho, \ldots, [\![s_n]\!]_\rho)$ where $\tau = \mathrm{type}(s_0)$. An equation $l \approx r \in \mathrm{Eqn}(\Sigma, X)$ is *valid* on $\mathcal{A}$ (denoted by $\mathcal{A} \models l \approx r$) if $[\![l]\!]_\rho = [\![r]\!]_\rho$ for all environments ρ for $\mathcal{A}$. A $\langle T, \Sigma\rangle$-theory E is said to be valid on $\mathcal{A}$ or $\mathcal{A}$ is a *model* of E (denoted by $\mathcal{A} \models E$) if all equations in E are valid on $\mathcal{A}$. An equation $l \approx r \in \mathrm{Eqn}(\Sigma, X)$ is a *theorem of* E or *equational consequence of* E (denoted by $E \models l \approx r$) if $l \approx r$ is valid on every model of E. An equivalence relation $\sim$ on $|\mathcal{A}|$ is said to be a *congruence* on $\mathcal{A}$ if (1) $a \sim b$ implies $a, b \in A^\tau$ for some $\tau \in T$ and (2) $a_0 \sim b_0, a_1 \sim b_1, \ldots, a_n \sim b_n$ implies $ap^\tau(a_0, a_1, \ldots, a_n) \sim ap^\tau(b_0, b_1, \ldots, b_n)$ for any $a_0, b_0 \in A^\tau$, $a_i, b_i \in A^{\tau_i}$ $(1 \leq i \leq n)$ where $\tau = \tau_1 \times \cdots \times \tau_n \to \tau_0$. We denote the $\sim$-equivalence class containing $a \in |\mathcal{A}|$ by $[a]$ i.e. $[a] = \{b \in |\mathcal{A}| \mid a \sim b\}$. The *quotient algebra* $\mathcal{A}/{\sim}$ has the carrier set $\bigcup_\tau (A/{\sim})^\tau$ where $(A/{\sim})^\tau = \{[a] \mid a \in A^\tau\}$, operations $ap^\tau_{\mathcal{A}/\sim}([a_0], [a_1], \ldots, [a_n]) = [ap^\tau_{\mathcal{A}}(a_0, a_1, \ldots, a_n)]$ and $c^{\mathcal{A}/\sim} = [c^{\mathcal{A}}]$ for each $c \in \Sigma$. It is readily checked that for a given $\langle T, \Sigma\rangle$-algebra $\mathcal{A}$ and a congruence $\sim$ on $\mathcal{A}$, the quotient algebra $\mathcal{A}/{\sim}$ is again a $\langle T, \Sigma\rangle$-algebra.

The following lemma will be used later.

$$\frac{l \approx r \in E}{l \approx r}\ \text{ax.} \qquad \frac{}{s \approx s}\ \text{refl.} \qquad \frac{t \approx s}{s \approx t}\ \text{sym.}$$

$$\frac{s \approx t \quad t \approx u}{s \approx u}\ \text{trans.} \qquad \frac{s_0 \approx t_0 \quad \cdots \quad s_n \approx t_n}{(s_0 \cdots s_n) \approx (t_0 \cdots t_n)}\ \text{cong.}$$

$$\frac{s \approx t}{s\theta \approx t\theta}\ \text{subst.} \qquad \frac{(s\ x_1 \cdots x_n) \approx (t\ x_1 \cdots x_n)}{s \approx t}\ \begin{array}{l}\text{ext.}\\ x_1, \ldots, x_n \notin \mathrm{V}(s) \cup \mathrm{V}(t)\end{array}$$

Figure 2 Inference rules for $E \vdash_{\text{ext}}$

▶ **Lemma 3.2.** Let E be a $\langle T, \Sigma\rangle$-theory, $\mathcal{A}$ a $\langle T, \Sigma\rangle$-algebra and X the set of variables over T. Then $[\![s\theta]\!]_\rho = [\![s]\!]_{\rho/\theta}$ holds for any S-expression $s \in \mathrm{S}(\Sigma, X)$, environment ρ for $\mathcal{A}$ and substitution $\theta : X \to \mathrm{S}(\Sigma, X)$. The environment ρ/θ is defined as: $(\rho/\theta)(x) = [\![\theta(x)]\!]_\rho$.

We next introduce a characterization of $\langle T, \Sigma\rangle$-algebras that incorporates extensionality to the semantics.

▶ **Definition 3.3** (extensional algebras and theorems)**.** Let $\mathcal{A} = \langle (A^\tau)_{\tau\in T}, (ap^\tau)_{\tau\in T^{\mathrm{f}}}, (c^{\mathcal{A}})_{c\in\Sigma}\rangle$ be a $\langle T, \Sigma\rangle$-algebra. Then $\mathcal{A}$ is said to be *extensional* if for all $\tau = \tau_1 \times \cdots \times \tau_n \to \tau_0 \in T^{\mathrm{f}}$ and $a_0, b_0 \in A^\tau$, $a_0 = b_0$ holds whenever $ap^\tau(a_0, a_1, \ldots, a_n) = ap^\tau(b_0, a_1, \ldots, a_n)$ for all $a_1 \in A^{\tau_1}, \ldots, a_n \in A^{\tau_n}$. An equation $l \approx r \in \mathrm{Eqn}(\Sigma, X)$ where X is the set of variables over T is said to be an *extensional theorem* (written as $E \models_{\text{ext}} l \approx r$) if $\mathcal{A} \models E$ implies $\mathcal{A} \models l \approx r$ for every extensional $\langle T, \Sigma\rangle$-algebra $\mathcal{A}$.

Let $\mathcal{A}$ be an extensional $\langle T, \Sigma\rangle$-algebra where $\mathcal{A} = \langle (A^\tau)_{\tau\in T}, (ap^\tau)_{\tau\in T^{\mathrm{f}}}, (c^{\mathcal{A}})_{c\in\Sigma}\rangle$. A congruence $\sim$ on $\mathcal{A}$ is said to be *extensional* if $ap^\tau(a_0, a_1, \ldots, a_n) \sim ap^\tau(b_0, a_1, \ldots, a_n)$ for all $a_1 \in A^{\tau_1}, \ldots, a_n \in A^{\tau_n}$ implies $a_0 \sim b_0$, for all $a_0, b_0 \in A^\tau$ where $\tau = \tau_1 \times \cdots \times \tau_n \to \tau_0$. It is straightforward to show that the quotient algebra $\mathcal{A}/\!\sim$ is an extensional $\langle T, \Sigma\rangle$-algebra if $\sim$ is an extensional congruence on $\mathcal{A}$.

The syntactic counterpart of extensional theorems is given as follows.

▶ **Definition 3.4** (extensional equational deduction)**.** Let E be a $\langle T, \Sigma\rangle$-theory and X the set of variables over T. The inference rules of *extensional equational deduction* are given in Figure 2. We write $E \vdash_{\text{ext}} s \approx t$ if $s \approx t \in \mathrm{Eqn}(\Sigma, X)$ is derivable by extensional equational deduction.

It is easy to see that $E \vdash_{\text{ext}} s \approx t$ if and only if $E \vdash_{\text{ext}} s\!\uparrow \approx t\!\uparrow$.

Our next aim is to develop the completeness theorem for extensional equational deduction (w.r.t. extensional theorems). For this, we need a couple of preparations.

Let E be a $\langle T, \Sigma\rangle$-theory and X the set of variables over T. The *extensional equivalence relation* $\overset{\text{ext}*}{\leftrightarrow}_E$ of E on $\mathrm{S}(\Sigma, X)$ is the smallest equivalence relation satisfying (1) $l \approx r \in E$ implies $l\theta \overset{\text{ext}*}{\leftrightarrow}_E r\theta$ for all substitutions θ, (2) $(s\ x_1 \cdots x_n) \overset{\text{ext}*}{\leftrightarrow}_E (t\ x_1 \cdots x_n)$ implies $s \overset{\text{ext}*}{\leftrightarrow}_E t$ where $x_1, \ldots, x_n \notin \mathrm{V}(s) \cup \mathrm{V}(t)$ and (3) $s_i \overset{\text{ext}*}{\leftrightarrow}_E t_i$ for all $0 \leq i \leq n$ implies $(s_0 \cdots s_n) \overset{\text{ext}*}{\leftrightarrow}_E (t_0 \cdots t_n)$. It is easy to see that $E \vdash_{\text{ext}} s \approx t$ if and only if $s \overset{\text{ext}*}{\leftrightarrow}_E t$. From here on, we assume[2] that $\mathrm{S}(\Sigma)^\tau \neq \emptyset$ for any $\tau \in T$.

[2] This assumption is required to guarantee the carrier sets of the term algebras satisfy the non-emptiness condition.

Let E be a $\langle T, \Sigma\rangle$-theory, X the set of variables over T, and Y a set of variables such that $Y \subseteq X$. A $\langle T, \Sigma\rangle$-algebra given by

$$\mathcal{T}_\Sigma(Y) = \langle (\mathrm{S}(\Sigma, Y)^\tau)_{\tau \in T}, (ap^\tau)_{\tau \in T^{\mathrm{f}}}, (c^{\mathcal{T}_\Sigma(Y)})_{c \in \Sigma}\rangle$$

where ap^τ and $c^{\mathcal{T}_\Sigma(Y)}$ are defined by $ap^\tau(s_0, s_1, \ldots, s_n) = (s_0\ s_1 \cdots s_n)$ and $c^{\mathcal{T}_\Sigma(Y)} = c$ is called a *$\langle T, \Sigma\rangle$-term algebra (with the set Y of generators)*. Note that by our assumption that $\mathrm{S}(\Sigma)^\tau \neq \emptyset$ for any $\tau \in T$, $\mathrm{S}(\Sigma, Y)^\tau \neq \emptyset$ for any set $Y \subseteq X$ and hence any $\langle T, \Sigma\rangle$-term algebra is a $\langle T, \Sigma\rangle$-algebra. It is not difficult to show that $\overset{\mathrm{ext}*}{\leftrightarrow}_E$ is an extensional congruence on the $\langle T, \Sigma\rangle$-term algebra $\mathcal{T}_\Sigma(X)$. If the set of generators is an arbitrary $Y \subseteq X$, however, then $\overset{\mathrm{ext}*}{\leftrightarrow}_E$ may not be an extensional congruence on $\mathcal{T}_\Sigma(Y)$ and cannot be used to define the initial extensional $\langle T, \Sigma\rangle$-algebra. To overcome this, Meinke [17] introduced an ω-evaluation rule. Here we use the following equivalence relation $\overset{\mathrm{ext}*}{\leftrightarrow}_{E,\omega}$. Let E be a $\langle T, \Sigma\rangle$-theory and Y a set of variables over T. The *ω-extensional equivalence relation* $\overset{\mathrm{ext}*}{\leftrightarrow}_{E,\omega}$ of E on $\mathrm{S}(\Sigma, Y)$ is obtained by replacing condition (2) in the definition of $\overset{\mathrm{ext}*}{\leftrightarrow}_E$ by (2′) $(s\ u_1 \cdots u_n) \overset{\mathrm{ext}*}{\leftrightarrow}_{E,\omega} (t\ u_1 \cdots u_n)$ for any $u_1 \in \mathrm{S}(\Sigma, Y)^{\tau_1}, \ldots, u_n \in \mathrm{S}(\Sigma, Y)^{\tau_n}$ implies $s \overset{\mathrm{ext}*}{\leftrightarrow}_{E,\omega} t$, where $\mathrm{type}(s) = \mathrm{type}(t) = \tau_1 \times \cdots \times \tau_n \to \tau_0$. Then $\overset{\mathrm{ext}*}{\leftrightarrow}_{E,\omega}$ is an extensional congruence on any $\langle T, \Sigma\rangle$-term algebra $\mathcal{T}_\Sigma(Y)$. Hence we get an extensional $\langle T, \Sigma\rangle$-algebra $\mathcal{T}_E(Y) = \mathcal{T}_\Sigma(Y)/\overset{\mathrm{ext}*}{\leftrightarrow}_{E,\omega}$. It is easy to see that for any $s{\uparrow} \approx t{\uparrow} \in \mathrm{Eqn}(\Sigma, Y)$, $\mathcal{T}_E(Y) \models s \approx t$ if and only if $\mathcal{T}_E(Y) \models s{\uparrow} \approx t{\uparrow}$.

Using standard arguments [5], the soundness and completeness of extensional equational deduction can be shown [17].

▶ **Theorem 3.5** (soundness and completeness [17])**.** Let E be a $\langle T, \Sigma\rangle$-theory and X the set of variables over T. For any $l \approx r \in \mathrm{Eqn}(\Sigma, X)$, $E \vdash_{\mathrm{ext}} l \approx r$ if and only if $E \models_{\mathrm{ext}} l \approx r$.

Our extensional semantics naturally leads to the notion of extensional inductive theorems.

▶ **Definition 3.6** (extensional inductive theorem [17])**.** Let E be a $\langle T, \Sigma\rangle$-theory and X the set of variables over T. An equation $s \approx t \in \mathrm{Eqn}(\Sigma, X)$ is said to be an *extensional inductive theorem* of E (denoted by $E \models_{\mathrm{eind}} s \approx t$) if $\mathcal{T}_E(\emptyset) \models s \approx t$.

The following characterization of extensional inductive theorems will be used later.

▶ **Lemma 3.7.** Let E be a $\langle T, \Sigma\rangle$-theory and X the set of variables over T. For any $s \approx t \in \mathrm{Eqn}(\Sigma, X)$, $E \models_{\mathrm{eind}} s \approx t$ if $s\theta \overset{\mathrm{ext}*}{\leftrightarrow}_E t\theta$ (on $\mathrm{S}(\Sigma, X)$) for any ground substitution $\theta : X \to \mathrm{S}(\Sigma)$.

4 Natural Semantics and Natural Inductive Theorems

Extensional semantics developed in the previous section and the notion of extensional inductive theorems introduced there seems to form a firm basis for simply typed equational theories. However, the notion of extensional inductive theorems lacks a property of inductive theorems in first-order term rewriting: *extensibility*, meaning that inductive theorems are preserved when a new function by a macro is added.

▶ **Example 4.1.** Let $T = \{\mathrm{Nat},\ \mathrm{Nat} \to \mathrm{Nat},\ \mathrm{Nat} \times \mathrm{Nat} \to \mathrm{Nat},\ (\mathrm{Nat} \to \mathrm{Nat}) \to \mathrm{Nat}\}$, $\Sigma = \{+^{\mathrm{Nat} \times \mathrm{Nat} \to \mathrm{Nat}}, \mathsf{0}^{\mathrm{Nat}}, \mathsf{s}^{\mathrm{Nat} \to \mathrm{Nat}}, \mathsf{zero}^{(\mathrm{Nat} \to \mathrm{Nat}) \to \mathrm{Nat}}\}$ and

$$E = \left\{ \begin{array}{lcl} +\ \mathsf{0}\ y & \approx & y \\ +\ (\mathsf{s}\ x)\ y & \approx & \mathsf{s}\ (+\ x\ y) \\ \mathsf{zero}\ \mathsf{s} & \approx & \mathsf{0} \end{array} \right\}.$$

Then $E \models_{\rm eind} \mathsf{zero}\ F \approx \mathsf{0}$. (For, the only possible instantiation of F is s.) By introducing a new constant $\mathsf{id}^{\rm Nat \to Nat}$, define $E' = \{\mathsf{id}\ x \approx x\} \cup E$. Then we do not have $E' \models_{\rm eind} \mathsf{zero}\ F \approx \mathsf{0}$ any more, since $\mathsf{zero\ id} \overset{\rm ext}{\leftrightarrow}{}^{*}_{E',\omega} \mathsf{0}$ does not hold.

From this example, it is observed that we may not conclude $\mathsf{zero}\ F \approx \mathsf{0}$ is an "inductive theorem" since this fact depends on the limited possibility of instantiating the variable F of function type. Hence this example suggests that the notion of extensional inductive theorems may be too general if validity needs to be preserved under addition of new function definitions. This motivates us to restrict extensional $\langle T, \Sigma\rangle$-algebras to *natural* $\langle T, \Sigma\rangle$-algebras.

▶ **Definition 4.2** (natural algebras). A $\langle T, \Sigma\rangle$-algebra $\mathcal{A} = \langle (A^\tau)_{\tau \in T}, (ap^\tau)_{\tau \in T^{\rm f}}, (c^{\mathcal{A}})_{c \in \Sigma}\rangle$ is said to be *natural* if for any $\tau \in T^{\rm f}$ with $\tau = \tau_1 \times \cdots \times \tau_n \to \tau_0$, (1) $A^\tau = [A^{\tau_1} \times \cdots \times A^{\tau_n} \to A^{\tau_0}]$ and (2) $ap^\tau(f, a_1, \ldots, a_n) = f(a_1, \ldots, a_n)$. Henceforth, a natural $\langle T, \Sigma\rangle$-algebra $\mathcal{A} = \langle (A^\tau)_{\tau \in T}, (ap^\tau)_{\tau \in T^{\rm f}}, (c^{\mathcal{A}})_{c \in \Sigma}\rangle$ is specified as $\mathcal{A} = \langle (A^\tau)_{\tau \in B}, (c^{\mathcal{A}})_{c \in \Sigma}\rangle$. A natural $\langle T, \Sigma\rangle$-algebra $\langle (A^\tau)_{\tau \in B}, (c^{\mathcal{A}})_{c \in \Sigma}\rangle$ is a *natural* $\langle T, \Sigma\rangle$*-term algebra (with the set X of generators)* if there exists $\Sigma' \subseteq \Sigma$ such that $A^\tau = \mathrm{S}(\Sigma', X)^\tau$ for each $\tau \in B$.

▶ **Example 4.3.** Consider T, Σ, and E from Example 4.1 and let $\Sigma' = \{\mathsf{0}, \mathsf{s}\}$. Let X be a set of variables over T and put $A^{\rm Nat} = \mathrm{S}(\Sigma', X)^{\rm Nat}$. Take $A^\tau = [A^{\tau_1} \times \cdots \times A^{\tau_n} \to A^{\tau_0}]$ for $\tau = \tau_1 \times \cdots \times \tau_n \to \tau_0 \in T^{\rm f}$ and any $c^{\mathcal{A}}$ for each $c^\tau \in \{\mathsf{0}, \mathsf{s}, +, \mathsf{zero}\}$ such that $c^{\mathcal{A}} \in A^\tau$. Then $\langle (A^\tau)_{\tau \in B}, (c^{\mathcal{A}})_{c \in \Sigma}\rangle$ is a natural $\langle T, \Sigma\rangle$-term algebra (with the set X of generators).

▶ **Lemma 4.4.** Any natural $\langle T, \Sigma\rangle$-algebra is extensional.

Proof. For any $f, g \in [A^{\tau_1} \times \cdots \times A^{\tau_n} \to A^{\tau_0}]$, $f = g$ iff $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$ holds for any $a_1 \in A^{\tau_1}, \ldots, a_n \in A^{\tau_n}$. ◀

Extensional inductive theorems were defined (in Definition 3.6) based on the $\langle T, \Sigma\rangle$-term algebra with the empty set of generators. Similarly, we will define a notion of *natural inductive theorems* from natural $\langle T, \Sigma\rangle$-term algebras with the empty set of generators. It is, however, not possible to directly relate the notion of natural inductive theorems to those of extensional theorems and extensional inductive theorems; we further require *consistency* of the natural $\langle T, \Sigma\rangle$-term algebras to connect these notions.

▶ **Definition 4.5** (natural inductive theorems). Let E be a $\langle T, \Sigma\rangle$-theory and X be the set of variables over T. Furthermore, assume that the set $\Sigma_{\rm c}$ of constructors is *free*, i.e. for any $s, t \in \mathrm{S}(\Sigma_{\rm c}, X)$ $s \overset{\rm ext}{\leftrightarrow}{}^{*}_{E} t$ implies $s = t$.

1. A natural $\langle T, \Sigma\rangle$-term algebra $\mathcal{A} = \langle (A^\tau)_{\tau \in B}, (c^{\mathcal{A}})_{c \in \Sigma}\rangle$ is said to be *consistent* with E if (1) $s \overset{\rm ext}{\leftrightarrow}{}^{*}_{E} [\![s]\!]$ holds for any $s \in \mathrm{S}(\Sigma)^{\rm b}$ and (2) $\mathcal{A} \models E$. Here, note that ρ of $[\![s]\!]_\rho$ can be safely omitted because $\mathrm{V}(s) = \emptyset$.
2. A natural $\langle T, \Sigma\rangle$-term algebra $\mathcal{A} = \langle (A)_{\tau \in B}, (c^{\mathcal{A}})_{c \in \Sigma}\rangle$ is said to be a *natural* $\langle T, \Sigma\rangle$*-term algebra for* E if $A^\tau = \mathrm{S}(\Sigma_{\rm c})^\tau$ for each $\tau \in B$ and $\mathcal{A}$ is consistent with E, where $\Sigma_{\rm c}$ is the set of constructors designated in E. E is said to be a *natural* $\langle T, \Sigma\rangle$*-theory* if there exists a natural $\langle T, \Sigma\rangle$-term algebra $\mathcal{A}$ for E.
3. Suppose that E is a natural $\langle T, \Sigma\rangle$-theory. Then an equation $l \approx r \in \mathrm{Eqn}(\Sigma, X)$ is said to be a *natural inductive theorem* of E (denoted by $E \models_{\rm nind} l \approx r$) if $\mathcal{A} \models l \approx r$ for any natural $\langle T, \Sigma\rangle$-term algebra $\mathcal{A}$ for E.

▶ **Example 4.6.** Consider T, Σ, and E from Example 4.1 and let $\Sigma_{\rm c} = \{\mathsf{0}, \mathsf{s}\}$. Put $A^{\rm Nat} = \mathrm{S}(\Sigma_{\rm c})^{\rm Nat}$ and $A^\tau = [A^{\tau_1} \times \cdots \times A^{\tau_n} \to A^{\tau_0}]$ for $\tau = \tau_1 \times \cdots \times \tau_n \to \tau_0 \in T^{\rm f}$. Let $\mathsf{0}^{\mathcal{A}} = \mathsf{0}$, $\mathsf{s}^{\mathcal{A}}(x) = (\mathsf{s}\ x)$, $+^{\mathcal{A}}(x, y)$ be the unique normal form of $(+\ x\ y)$ w.r.t. $\{l \to r \mid l \approx r \in E\}$, and $\mathsf{zero}^{\mathcal{A}}(f) = \mathsf{0}$. Then $\mathcal{A} = \langle (A^\tau)_{\tau\in B}, (c^{\mathcal{A}})_{c\in\Sigma}\rangle$ is a natural $\langle T,\Sigma\rangle$-term algebra for E. For example, $(+\ (\mathsf{s}\ \mathsf{0})\ (\mathsf{s}\ \mathsf{0})) \overset{\text{ext}*}{\leftrightarrow}_E (\mathsf{s}\ (\mathsf{s}\ \mathsf{0})) = [\![(+\ (\mathsf{s}\ \mathsf{0})\ (\mathsf{s}\ \mathsf{0}))]\!]$ holds for (1) and $[\![(+\ 0\ y)]\!]_\rho = \rho(y) = [\![y]\!]_\rho$ holds for any ρ for (2). One can also set $\mathsf{zero}^{\mathcal{A}}(succ) = \mathsf{0}$ for $succ \in [\mathrm{S}(\Sigma_\mathrm{c})^{\mathrm{Nat}} \to \mathrm{S}(\Sigma_\mathrm{c})^{\mathrm{Nat}}]$ such that $succ(x) = (\mathsf{s}\ x)$; $\mathsf{zero}^{\mathcal{A}}(f) = (\mathsf{s}\ \mathsf{0})$ otherwise, to obtain a natural $\langle T,\Sigma\rangle$-term algebra for E. This implies that an equation $(\mathsf{zero}\ F) \approx \mathsf{0}$ is not a natural inductive theorem of E. In contrast, interpretations $\mathsf{s}^{\mathcal{A}}$ and $+^{\mathcal{A}}$ are common to all natural $\langle T,\Sigma\rangle$-term algebras for E and hence it follows that an equation $(+\ x\ y) \approx (+\ y\ x)$ is a natural inductive theorem of E.

Our first aim is to show the relation of natural inductive theorems to extensional theorems and extensional inductive theorems.

▶ **Lemma 4.7.** Let E be a natural $\langle T,\Sigma\rangle$-theory. The set of natural inductive theorems of E is closed under the inference rules of Figure 2.

Proof. This follows from the fact that for any extensional $\langle T,\Sigma\rangle$-algebra $\mathcal{A}$ for E, the set $\mathrm{Th}(\mathcal{A}) = \{s \approx t \mid \mathcal{A} \models s \approx t\}$ is closed under the inference rules of Figure 2. ◀

It easily follows from this lemma that for any equation $l \approx r \in \mathrm{Eqn}(\Sigma, X)$, $E \models_{\mathrm{nind}} l \approx r$ iff $E \models_{\mathrm{nind}} l\!\uparrow \approx r\!\uparrow$.

▶ **Lemma 4.8.** Let $\mathcal{A} = \langle (A^\tau)_{\tau\in B}, (c^{\mathcal{A}})_{c\in\Sigma}\rangle$ be a natural $\langle T,\Sigma\rangle$-term algebra for E. For any $s, t \in \mathrm{S}(\Sigma)^{\mathrm{b}}$, $s \overset{\text{ext}*}{\leftrightarrow}_E t$ iff $[\![s]\!] = [\![t]\!]$.

Proof. Let $s, t \in \mathrm{S}(\Sigma)^{\mathrm{b}}$. By condition (1) of consistency, $[\![s]\!] \overset{\text{ext}*}{\leftrightarrow}_E s$ and $[\![t]\!] \overset{\text{ext}*}{\leftrightarrow}_E t$. Hence, $s \overset{\text{ext}*}{\leftrightarrow}_E t$ iff $[\![s]\!] \overset{\text{ext}*}{\leftrightarrow}_E [\![t]\!]$. Furthermore, since $[\![s]\!], [\![t]\!] \in \mathrm{S}(\Sigma_\mathrm{c})$, $[\![s]\!] \overset{\text{ext}*}{\leftrightarrow}_E [\![t]\!]$ iff $[\![s]\!] = [\![t]\!]$ by our assumption that Σ_c is free (Definition 4.5). ◀

We arrive at one of the main theorems of this section.

▶ **Theorem 4.9.** Let E be a natural $\langle T,\Sigma\rangle$-theory and X be the set of variables over T. For any $l \approx r \in \mathrm{Eqn}(\Sigma, X)$, (1) $E \models_{\mathrm{ext}} l \approx r$ implies $E \models_{\mathrm{nind}} l \approx r$; (2) $E \models_{\mathrm{nind}} l \approx r$ implies $E \models_{\mathrm{eind}} l \approx r$.

Proof. (1) By Lemma 4.7. (2) Since $E \models_{\mathrm{nind}} l \approx r$ iff $E \models_{\mathrm{nind}} l\!\uparrow \approx r\!\uparrow$ holds and $E \models_{\mathrm{eind}} l \approx r$ iff $E \models_{\mathrm{eind}} l\!\uparrow \approx r\!\uparrow$ holds, w.l.o.g. we assume that l, r have a base type. If $l \approx r$ is a natural inductive theorem of E then so is $l\theta \approx r\theta$ for any ground substitution θ by Lemma 4.7. Thus $[\![l\theta]\!] = [\![r\theta]\!]$ for any natural $\langle T,\Sigma\rangle$-term algebra $\mathcal{A}$ for E. Then by Lemma 4.8, $l\theta \overset{\text{ext}*}{\leftrightarrow}_E r\theta$. Thus, by Lemma 3.7, $E \models_{\mathrm{eind}} l \approx r$. ◀

Our next aim is to show extensibility of natural inductive theorems. We introduce two new conditions for this.

▶ **Definition 4.10** (constructor-based theories)**.** A $\langle T,\Sigma\rangle$-theory E is said to be *constructor-based* if for any $f \in \Sigma_\mathrm{d}$ and any substitution $\sigma : \mathrm{V}^{\mathrm{b}}(f\!\uparrow) \to \mathrm{S}(\Sigma_\mathrm{c})$, there exists $t \in \mathrm{S}(\Sigma_\mathrm{c}, \mathrm{V}^{\mathrm{f}}(f\!\uparrow))$ such that $(f\!\uparrow)\sigma \overset{\text{ext}*}{\leftrightarrow}_E t$.

▶ **Definition 4.11** (conservative extensions)**.** Let E be a $\langle T,\Sigma\rangle$-theory and E' a Σ'-theory such that $E \subseteq E'$ and $\Sigma \subseteq \Sigma'$. Then E' is said to be a *conservative extension of* E if (1) $\Sigma_\mathrm{c} = \Sigma'_\mathrm{c}$ and (2) for all S-expressions $s, t \in \mathrm{S}(\Sigma_\mathrm{c})$, $s \overset{\text{ext}*}{\leftrightarrow}_E t$ iff $s \overset{\text{ext}*}{\leftrightarrow}_{E'} t$.

We introduce a saturated set for E to prove a property of elementary constructor-based theories (Lemma 4.16).

▶ **Definition 4.12** (saturated sets for E). Let E be a $\langle T, \Sigma\rangle$-theory. We define a set W^τ for each $\tau \in T$ like this: $W^\tau = \{s \in \mathrm{S}(\Sigma)^\tau \mid \exists t \in \mathrm{S}(\Sigma_\mathrm{c})\ s \overset{\mathrm{ext}}{\leftrightarrow}{}^*_E t\}$ for $\tau \in B$; $W^\tau = \{s_0 \in \mathrm{S}(\Sigma)^\tau \mid \forall s_1 \in W^{\tau_1} \cdots \forall s_n \in W^{\tau_n}\ (s_0\ s_1 \cdots s_n) \in W^{\tau_0}\}$ if $\tau = \tau_1 \times \cdots \times \tau_n \to \tau_0 \in T^\mathrm{f}$. The *saturated set* W for E is given by $W = \bigcup_{\tau \in T} W^\tau$.

▶ **Lemma 4.13.** Let W be the saturated set for a $\langle T, \Sigma\rangle$-theory E. (1) For any S-expression $s \in \mathrm{S}(\Sigma)$, $s \in W$ iff $(s{\uparrow})\sigma \in W$ for any substitution $\sigma : \mathrm{V}(s{\uparrow}) \to W$. (2) If $s \in W$ and $s_g \overset{\mathrm{ext}}{\leftrightarrow}{}^*_E t$ then $t \in W$.

Proof. (1) By definition. (2) By (1). ◀

▶ **Lemma 4.14.** Let E be an elementary $\langle T, \Sigma\rangle$-theory, X the set of variables over T and W the saturated set for E. Then $s\theta \in W$ holds for any S-expression $s \in \mathrm{S}(\Sigma_\mathrm{c}, X)$ and substitution $\theta : X \to W$.

Proof. By induction on s. ◀

▶ **Lemma 4.15.** Let E be an elementary constructor-based $\langle T, \Sigma\rangle$-theory and W the saturated set for E. Then $\mathrm{S}(\Sigma) = W$.

Proof. Use Lemmas 4.13 and 4.14 to show that $(s{\uparrow})\sigma \in W$ for every substitution $\sigma : \mathrm{V}(s{\uparrow}) \to W$ by induction on $s \in \mathrm{S}(\Sigma)$. ◀

The next lemma follows immediately from Lemma 4.15.

▶ **Lemma 4.16.** If E is an elementary constructor-based $\langle T, \Sigma\rangle$-theory then for any S-expression $s \in \mathrm{S}(\Sigma)^\mathrm{b}$ there exists an S-expression $t \in \mathrm{S}(\Sigma_\mathrm{c})^\mathrm{b}$ such that $s \overset{\mathrm{ext}}{\leftrightarrow}{}^*_E t$.

We arrive at the other main theorem of this section.

▶ **Theorem 4.17** (extensibility of natural inductive theorems). Let E be an elementary constructor-based natural $\langle T, \Sigma\rangle$-theory, X be the set of variables over T, f a new defined symbol of second-order type τ such that $f \notin \Sigma$ and $r \in \mathrm{S}(\Sigma, \{x_1, \ldots, x_n\})$ where $x_1, \ldots, x_n \in X^\mathrm{b}$. Let $T' = T \cup \{\tau\}$, $\Sigma' = \Sigma \cup \{f\}$ and suppose $E' = E \cup \{f\ x_1 \cdots x_n \approx r\}$ is a conservative extension of E. Then the following hold. (1) E' is an elementary constructor-based natural $\langle T', \Sigma'\rangle$-theory. (2) For any $s \approx t \in \mathrm{Eqn}(\Sigma, X)$, $E \models_\mathrm{nind} s \approx t$ iff $E' \models_\mathrm{nind} s \approx t$.

Proof. We first show ($\Rightarrow$) of (2). Suppose that there exists a natural $\langle T', \Sigma'\rangle$-term algebra $\mathcal{A}' = \langle (A'^\tau)_{\tau \in B}, (c^{\mathcal{A}'})_{c \in \Sigma'}\rangle$ for E' such that $s \approx t$ does not hold. By just omitting $f^{\mathcal{A}'}$ (and A'^τ for $\tau \in T' \backslash T$), we obtain a natural $\langle T, \Sigma\rangle$-term algebra $\mathcal{A}$ for E such that $s \approx t$ does not hold. Next we show (1) and ($\Leftarrow$) of (2). Since E is elementary and $\Sigma_\mathrm{c} = \Sigma'_\mathrm{c}$, E' is elementary. Let θ be a substitution such that $\theta : X^\mathrm{b} \to \mathrm{S}(\Sigma_\mathrm{c})$. Then, $(f{\uparrow})\theta \overset{\mathrm{ext}}{\leftrightarrow}{}^*_{E'} (r{\uparrow})\theta$. Furthermore, by Lemma 4.16, there exists $t \in \mathrm{S}(\Sigma_\mathrm{c})$ such that $(r{\uparrow})\theta \overset{\mathrm{ext}}{\leftrightarrow}{}^*_E t$. Thus $(f{\uparrow})\theta \overset{\mathrm{ext}}{\leftrightarrow}{}^*_{E'} t$. Hence E' is constructor-based. It remains to show that E' is natural. Since E is a natural $\langle T, \Sigma\rangle$-theory, there exists a natural $\langle T, \Sigma\rangle$-term algebra $\mathcal{A}$ for E. Let $\mathcal{A} = \langle (A^\tau)_{\tau \in B}, (c^\mathcal{A})_{c \in \Sigma}\rangle$. Then, by the definition, (1) for any $s \in \mathrm{S}(\Sigma)^\mathrm{b}$, $[\![s]\!] \overset{\mathrm{ext}}{\leftrightarrow}{}^*_E s$ holds, and (2) for any $l \approx r \in E$ and for any environment ρ for $\mathcal{A}$, $[\![l]\!]_\rho = [\![r]\!]_\rho$ holds. We define a natural $\langle T', \Sigma'\rangle$-term algebra $\mathcal{A}'$ by $\mathcal{A}' = \langle (A'^\tau)_{\tau \in B}, (c^{\mathcal{A}'})_{c \in \Sigma'}\rangle$ where $A'^\tau = A^\tau$ for any $\tau \in B$, $c^{\mathcal{A}'} = c^\mathcal{A}$ for all $c \in \Sigma$ and $f^{\mathcal{A}'}(a_1, \ldots, a_n) = [\![r]\!]_\rho$ where $\rho = \{x_i \mapsto a_i \mid 1 \leq i \leq n\}$. We now show that $\mathcal{A}'$ is a natural $\langle T', \Sigma'\rangle$-term algebra for E'.

- $[\![s]\!] \overset{\text{ext}*}{\leftrightarrow}_{E'} s$ holds for any $s \in \mathrm{S}(\Sigma')^{\mathrm{b}}$: Let $s \in \mathrm{S}(\Sigma')^{\mathrm{b}}$. Then, since E' is elementary constructor-based, by Lemma 4.16 there exists an S-expression $u \in \mathrm{S}(\Sigma_{\mathrm{c}})$ such that $s \overset{\text{ext}*}{\leftrightarrow}_{E'} u$. Then $[\![s]\!] = [\![u]\!]$ by Lemma 4.8. Furthermore, $u = [\![u]\!]$ by $u \in \mathrm{S}(\Sigma_{\mathrm{c}})$. Therefore, $[\![s]\!] = [\![u]\!] = u_g \overset{\text{ext}*}{\leftrightarrow}_{E'} s$.
- For any $l \approx r \in E'$ and for any environment ρ for $\mathcal{A}'$, $[\![l]\!]_\rho = [\![r]\!]_\rho$ holds: this follows from the assumption and the definition of $f^{\mathcal{A}'}$.

Thus the proof of (1) has been completed. To show ($\Leftarrow$) of (2), suppose that $s \approx t$ does not hold in $\mathcal{A}$. Then, by construction, $s \approx t$ does not hold in $\mathcal{A}'$ either. ◀

▶ **Example 4.18.** If we drop the condition that f has a second-order type, then E' is not constructor-based in general. Let $T = \{\mathrm{Nat}, \mathrm{Nat} \to \mathrm{Nat}, \mathrm{Nat} \times \mathrm{Nat} \to \mathrm{Nat}\}$, $\Sigma_{\mathrm{d}} = \{+^{\mathrm{Nat}\times\mathrm{Nat}\to\mathrm{Nat}}\}$, $\Sigma_{\mathrm{c}} = \{\mathsf{s}^{\mathrm{Nat}\to\mathrm{Nat}}, \mathsf{0}^{\mathrm{Nat}}\}$ and

$$E = \left\{ \begin{array}{lcl} +\ \mathsf{0}\ y & \approx & y \\ +\ (\mathsf{s}\ x)\ y & \approx & \mathsf{s}\ (+\ x\ y) \end{array} \right\}.$$

Then E is an elementary constructor-based $\langle T, \Sigma\rangle$-theory. Take $E' = E \cup \{\mathsf{g}\ F\ x \approx +\ (F\ x)\ x\}$. Then there exists no $t \in \mathrm{S}(\Sigma_{\mathrm{c}}, \{F\})$ such that $\mathsf{g}\ F\ \mathsf{0} \overset{\text{ext}*}{\leftrightarrow}_{E'} t$.

5 Checking Natural Inductive Theorems

In this section, we present partial answers to the following questions:

1. When can one prove or check an equational theory is constructor-based?
2. When can one prove or check an equation is a natural inductive theorem?

We first answer the second question by giving a sufficient condition for natural inductive theorems.

▶ **Lemma 5.1.** Let E be a natural $\langle T, \Sigma\rangle$-theory, X the set of variables over T and $\mathcal{A}$ a natural $\langle T, \Sigma\rangle$-term algebra for E. Then for any environment ρ for $\mathcal{A}$, there exists a substitution $\sigma_g : X^{\mathrm{b}} \to \mathrm{S}(\Sigma_{\mathrm{c}})$ such that $\rho = \rho{\restriction}_{X^{\mathrm{f}}}/\sigma$.

Proof. Take a substitution $\sigma = \{x \mapsto \rho(x) \mid x \in X^{\mathrm{b}}\}$. We now show $\rho = (\rho{\restriction}_{X^{\mathrm{f}}})/\sigma$. For $x \in X^{\mathrm{b}}$, we have $((\rho{\restriction}_{X^{\mathrm{f}}})/\sigma)(x) = [\![\sigma(x)]\!]_{\rho\restriction_{X^{\mathrm{f}}}} = [\![\rho(x)]\!]_{\rho\restriction_{X^{\mathrm{f}}}} = \rho(x)$. For $F \in X^{\mathrm{f}}$, we have $((\rho{\restriction}_{X^{\mathrm{f}}})/\sigma)(F) = [\![\sigma(F)]\!]_{\rho\restriction_{X^{\mathrm{f}}}} = [\![F]\!]_{\rho\restriction_{X^{\mathrm{f}}}} = \rho(F)$. ◀

▶ **Theorem 5.2** (sufficient condition for natural inductive theorem)**.** Let E be a natural $\langle T, \Sigma\rangle$-theory, X the set of variables over T and $s \approx t \in \mathrm{Eqn}(\Sigma, X)$. If $s\sigma \overset{\text{ext}*}{\leftrightarrow}_{E} t\sigma$ for any substitution $\sigma : X^{\mathrm{b}} \to \mathrm{S}(\Sigma_{\mathrm{c}})$, then $s \approx t$ is a natural inductive theorem of E.

Proof. Let $\mathcal{A}$ be a natural $\langle T, \Sigma\rangle$-term algebra for E and ρ an environment for $\mathcal{A}$. By Lemma 5.1, there exists $\sigma : X^{\mathrm{b}} \to \mathrm{S}(\Sigma_{\mathrm{c}})$ such that $\rho = \rho{\restriction}_{X^{\mathrm{f}}}/\sigma$. By Lemma 3.2, Then $[\![u\sigma]\!]_{\rho\restriction_{X^{\mathrm{f}}}} = [\![u]\!]_{\rho\restriction_{X^{\mathrm{f}}}/\sigma} = [\![u]\!]_\rho$ for any $u \in \mathrm{S}(\Sigma, X)$. Thus $[\![s]\!]_\rho = [\![t]\!]_\rho$ iff $[\![s\sigma]\!]_{\rho\restriction_{X^{\mathrm{f}}}} = [\![t\sigma]\!]_{\rho\restriction_{X^{\mathrm{f}}}}$. By our assumption, $s\sigma \overset{\text{ext}*}{\leftrightarrow}_{E} t\sigma$. By Proposition 3.5, $E \models_{\mathrm{ext}} s\sigma \approx t\sigma$ holds. By our assumption, $\mathcal{A} \models E$. Thus, since $\mathcal{A}$ is an extensional $\langle T, \Sigma\rangle$-algebra by Lemma 4.4, $\mathcal{A} \models s\sigma \approx t\sigma$ holds. Hence for any environment ρ' for $\mathcal{A}$, $[\![s\sigma]\!]_{\rho'} = [\![t\sigma]\!]_{\rho'}$ and thus, in particular, $[\![s\sigma]\!]_{\rho\restriction_{X^{\mathrm{f}}}} = [\![t\sigma]\!]_{\rho\restriction_{X^{\mathrm{f}}}}$. This concludes $[\![s]\!]_\rho = [\![t]\!]_\rho$. ◀

We next answer the first question by giving a sufficient condition of equational theories (specified by STSRSs) to be constructor-based.

▶ **Definition 5.3** (simple S-expressions). An S-expression s is said to be *simple* if for all $(u\ t_1 \cdots t_n) \trianglelefteq s$, (1) if $\mathrm{head}(u) \in V$ then $\mathrm{type}(t_i) \in B$ for $i = 1, \ldots, n$ and (2) if $\mathrm{head}(u) \in \Sigma_\mathrm{d}$ and $\mathrm{type}(t_i) \in B$ then $t_i \in \mathrm{S}(\Sigma)$ for $i = 1, \ldots, n$.

▶ **Lemma 5.4.** Let $f \in \Sigma_\mathrm{d}$. For any substitution $\theta : V^\mathrm{b} \to \mathrm{S}(\Sigma)$, $(f\!\uparrow)\theta$ is simple.

Proof. Let $(f\!\uparrow)\theta = ((\cdots (f\ t_{11} \cdots t_{1n_1}) \cdots)\ t_{m1} \cdots t_{mn_m})$. By our assumption, if $\mathrm{type}(t_{ij}) \in B$ then $t_{ij} \in \mathrm{S}(\Sigma)$ and if $\mathrm{type}(t_{ij}) \notin B$ then $t_{ij} \in V^\mathrm{f}$. Thus for any $(u\ s_1 \cdots s_n) \trianglelefteq t_{ij} \in \mathrm{S}(\Sigma)$, the condition (1) holds since $\mathrm{head}(u) \notin V$ and the condition (2) holds since $s_1, \ldots, s_n \in \mathrm{S}(\Sigma)$. Furthermore, if $t_{ij} \in V^\mathrm{f}$ then $(u\ s_1 \cdots s_n) \trianglelefteq t_{ij}$ does not happen. Thus it remains to show the conditions (1) and (2) hold for $(u\ t_{k1} \cdots t_{kn_k})$ where $\mathrm{head}(u) = f$ and $1 \leq k \leq m$. The condition (1) holds since $f \notin V$. The condition (2) holds since $\mathrm{type}(t_{ki}) \in B$ implies $t_{ki} \in \mathrm{S}(\Sigma)$ for $i = 1, \ldots, n_k$. ◀

An S-expression s such that $s \to_\mathcal{R} t$ for no t is said to be *normal*; the set of normal S-expressions of $\mathcal{R}$ is denoted by $\mathrm{NF}(\mathcal{R})$. An STSRS $\mathcal{R}$ is said to be *higher-order quasi-reducible* (denoted by $\mathrm{HQR}(\mathcal{R})$) if $s \notin \mathrm{NF}(\mathcal{R})$ for any S-expression $s \in \mathrm{S}(\Sigma, V^\mathrm{f})^\mathrm{b}$ such that (i) $\mathrm{head}(s) \in \Sigma_\mathrm{d}$ and (ii) for any $u \in \mathrm{Args}(s)$, if $\mathrm{type}(u) \in B$ then $u \in \mathrm{S}(\Sigma_\mathrm{c})$ and otherwise $u \in V^\mathrm{f}$ [4].

▶ **Lemma 5.5.** Let $\mathcal{R}$ be a left-linear elementary STSRS such that $\mathrm{HQR}(\mathcal{R})$ hold. Let $s \in \mathrm{S}(\Sigma, V^\mathrm{f})^\mathrm{b} \cap \mathrm{NF}(\mathcal{R})$. If s is simple then $s \in \mathrm{S}(\Sigma_\mathrm{c}, V^\mathrm{f})$.

Proof. Take a minimal (w.r.t. subexpression relation $\trianglelefteq$) $s \in \mathrm{S}(\Sigma, V^\mathrm{f})^\mathrm{b} \cap \mathrm{NF}(\mathcal{R})$ such that s is simple and $s \notin \mathrm{S}(\Sigma_\mathrm{c}, V^\mathrm{f})$. Then there exists a subexpression u of s such that $\mathrm{head}(u) = f \in \Sigma_\mathrm{d}$. Take a maximal (w.r.t. subexpression relation $\trianglelefteq$) such subexpression u. We first claim that $\mathrm{type}(u) \in B$. Suppose $\mathrm{type}(u) \notin B$. Then since $\mathrm{type}(s) \in B$, there exists a subexpression $(u_0 \cdots u \cdots)$ of s. If $\mathrm{head}(u_0) \in V$ then, since s is simple, $\mathrm{type}(u) \in B$ and hence this contradicts our assumption. Otherwise by the maximality of u, $\mathrm{head}(u_0) \in \Sigma_\mathrm{c}$. Then, since $\mathcal{R}$ is elementary, it follows $\mathrm{type}(u) \in B$. Hence this also contradicts our assumption. Therefore $\mathrm{type}(u) \in B$. Let $u = ((\cdots (f\ t_{11} \cdots t_{1n_1}) \cdots)\ t_{m1} \cdots t_{mn_m})$. Since s is simple, if $\mathrm{type}(t_{ij}) \in B$ then $t_{ij} \in \mathrm{S}(\Sigma)$; furthermore, by $t_{ij} \trianglelefteq s$, $t_{ij} \in \mathrm{NF}(\mathcal{R})$ and simple. Thus by the minimality of s, $\mathrm{type}(t_{ij}) \in B$ implies $t_{ij} \in \mathrm{S}(\Sigma_\mathrm{c})$. Then, since $\mathcal{R}$ is left-linear and higher-order quasi-reducible and $\mathrm{type}(u) \in B$, $u \notin \mathrm{NF}(\mathcal{R})$. This is a contradiction. ◀

▶ **Definition 5.6** (GAV/quasi-simple/simplicity-preserving). **1.** The set $\mathrm{GAV}(s)$ of *ground-augmenting variables* of an S-expression s is defined like this: $\mathrm{GAV}(a) = \emptyset$ for $a \in \Sigma \cup V$; $\mathrm{GAV}((t_0\ t_1 \cdots t_n)) = (\bigcup\{\mathrm{GAV}(t_i) \mid 0 \leq i \leq n\}) \cup (\bigcup\{\mathrm{V}(t_i) \mid \mathrm{type}(t_i) \in B, 1 \leq i \leq n, \mathrm{head}(t_0) \in \Sigma_\mathrm{d}\})$.

2. An S-expression s is said to be *quasi-simple w.r.t. a set* X of variables if for any subexpression $(u\ t_1 \cdots t_n)$ of s, (1) if $\mathrm{head}(u) \in V$ then $t_i \in \mathrm{S}(\Sigma, X)^\mathrm{b}$, and (2) if $\mathrm{head}(u) \in \Sigma_\mathrm{d}$ and $\mathrm{type}(t_i) \in B$ then $t_i \in \mathrm{S}(\Sigma, X)$.
3. A rewrite rule $l \to r$ of type τ is said to be *simplicity-preserving* if (1) $\mathrm{head}(r) \in V$ implies τ is second-order, (2) $\mathrm{head}(r) \notin \Sigma \cup \mathrm{GAV}(l)$ implies $\tau \in B$ and (3) r is quasi-simple w.r.t. $\mathrm{GAV}(l)$. An STSRS $\mathcal{R}$ is simplicity-preserving if it consists of simplicity-preserving rewrite rules.

Let $l \to r$ be a rewrite rule. Suppose $\mathrm{head}(l)$ has the second-order type. Then $\mathrm{GAV}(l) = \mathrm{V}(l)$. Hence if moreover $\mathrm{V}(r) \subseteq V^\mathrm{b}$ then r is quasi-simple w.r.t. $\mathrm{GAV}(l)$. Therefore, if moreover $l \to r$ has a base type then $l \to r$ is simplicity-preserving.

▶ **Example 5.7.** Let $T = \{\text{Nat}, \text{Nat} \to \text{Nat}, \text{Nat} \times \text{Nat} \to \text{Nat}, \text{List}, \text{Nat} \times \text{List} \to \text{List}, \text{List} \times \text{List} \to \text{List}, (\text{Nat} \to \text{Nat}) \times \text{List} \to \text{List}\}$, $\Sigma_c = \{0^{\text{Nat}}, \mathsf{s}^{\text{Nat}\to\text{Nat}}, []^{\text{List}}, :^{\text{Nat}\times\text{List}\to\text{List}}\}$, $\Sigma_d = \{+^{\text{Nat}\times\text{Nat}\to\text{Nat}}, \mathsf{app}^{\text{List}\times\text{List}\to\text{List}}, \mathsf{map}^{(\text{Nat}\to\text{Nat})\times\text{List}\to\text{List}}\}$ and

$$\mathcal{R} = \left\{ \begin{array}{llll} (1) & +\ 0\ y & \to & y \\ (2) & +\ (\mathsf{s}\ x)\ y & \to & \mathsf{s}\ (+\ x\ y) \\ (3) & \mathsf{app}\ []\ ys & \to & ys \\ (4) & \mathsf{app}\ (:\ x\ xs)\ ys & \to & :\ x\ (\mathsf{app}\ xs\ ys) \\ (5) & \mathsf{map}\ F\ [] & \to & [] \\ (6) & \mathsf{map}\ F\ (:\ x\ xs) & \to & :\ (F\ x)\ (\mathsf{map}\ F\ xs) \end{array} \right\}.$$

By the remark above, rules (1)–(4) are simplicity-preserving. Let $l = \mathsf{map}\ F\ []$ and $r = []$. Then $\text{GAV}(l) = \emptyset$. r is quasi-simple because there is no subexpression of the form $(u\ t_1 \cdots t_n)$. Hence, since $l \to r$ has a base type, $l \to r$ is simplicity-preserving. Let $l = \mathsf{map}\ F\ (:\ x\ xs)$ and $r = :\ (F\ x)\ (\mathsf{map}\ F\ xs)$. The set $\text{GAV}(l) = \{x, xs\}$. Let $X = \{x, xs\}$. For $(u\ t_1) = (F\ x)$, we have $t_1 = x \in \text{S}(\Sigma, X)^{\text{b}}$ and for $(u\ t_1\ t_2) = (\mathsf{map}\ F\ xs)$, we have $t_2 = xs \in \text{S}(\Sigma, X)^{\text{b}}$. Since $l \to r$ has a base type and r is quasi-simple w.r.t. $\text{GAV}(l)$, $l \to r$ is simplicity-preserving.

▶ **Lemma 5.8.** Let s be an S-expression and σ a substitution. If $s\sigma$ is simple then $\theta(x) \in \text{S}(\Sigma)$ for any $x \in \text{GAV}(s)$.

Proof. By induction on s. ◀

▶ **Lemma 5.9.** Let $\mathcal{R}$ be a simplicity-preserving STSRS. If s is simple and $s \to_{\mathcal{R}} t$ then t is simple.

Proof. Suppose $s = C[l\sigma]$, $t = C[r\sigma]$ and $l \to r$ is simplicity-preserving. We first show $r'\sigma$ is simple for any $r' \trianglelefteq r$, by induction on r'. The case of $r' \in \Sigma \cup V$ follows easily. Let $r' = (r_0\ r_1 \cdots r_n)$.

(1) Suppose $\text{head}(r_0\sigma) \in V$. Then $\text{head}(r_0) \in V$. Then $\text{type}(r_i) \in B$ for $1 \leq i \leq n$ by quasi-simplicity of r. Thus $\text{type}(r_i\sigma) \in B$ for $1 \leq i \leq n$.

(2) Suppose $\text{head}(r_0\sigma) \in \Sigma_d$. We distinguish two cases. Case $\text{head}(r_0) = \text{head}(r_0\sigma)$. By the quasi-simplicity of r w.r.t. $\text{GAV}(l)$, $\text{type}(r_i) \in B$ implies $r_i \in \text{S}(\Sigma, \text{GAV}(l))$ $(1 \leq i \leq n)$. Since $l\sigma$ is simple, $\sigma(x) \in \text{S}(\Sigma)$ for any $x \in \text{GAV}(l)$ by Lemma 5.8. Hence $\text{type}(r_i\sigma) \in B$ implies $r_i\sigma \in \text{S}(\Sigma)$. Case $\text{head}(r_0) \in V$. Then, by the quasi-simplicity of r w.r.t. $\text{GAV}(l)$, $r_i \in \text{S}(\Sigma, \text{GAV}(l))^{\text{b}}$ for $1 \leq i \leq n$. Again, by Lemma 5.8, it follows that $r_i\sigma \in \text{S}(\Sigma)$.

Hence we conclude that $r\sigma$ is simple. Next, we show $C'[r\sigma]$ is simple by induction on $C' \trianglelefteq C$ (B.S.) follows from the fact that $r\sigma$ is simple. To show (I.S.), we distinguish two cases.

- Case $C' = (C''\ w_1 \cdots w_n)$.
  (1) Suppose $\text{head}(C''[r\sigma]) \in V$. Case of $\text{head}(C''[l\sigma]) \in V$ is trivial. Suppose $\text{head}(C''[l\sigma]) \notin V$. Then $\text{head}(C''[r\sigma]) = \text{head}(r\sigma)$ and hence $\text{head}(r) \in V$. Thus, $\text{type}(l\sigma)$ is second-order and hence $\text{type}(w_i) \in B$ for all $1 \leq i \leq n$.
  (2) Suppose $\text{head}(C''[r\sigma]) \in \Sigma_d$. If $\text{head}(C''[r\sigma]) = \text{head}(C'')$, then $\text{head}(C''[l\sigma]) \in \Sigma_d$ and hence $\text{type}(w_i) \in B$ implies $w_i \in \text{S}(\Sigma)$. If $\text{head}(C''[r\sigma]) = \text{head}(r\sigma)$, then $\text{head}(C''[l\sigma]) \in \Sigma_d$. Thus $\text{type}(w_i) \in B$ implies $w_i \in \text{S}(\Sigma)$.
- Case $C' = (w_0 \cdots C'' \cdots)$. If $\text{head}(w_0) \in V$ then $\text{type}(C''[l\sigma]) \in B$ and hence $\text{type}(C''[r\sigma]) \in B$. If $\text{head}(w_0) \in \Sigma_d$ and $\text{type}(C''[r\sigma]) \in B$ then $C''[l\sigma] \in \text{S}(\Sigma)$ and hence $C''[r\sigma] \in \text{S}(\Sigma)$.

◀

An STSRS $\mathcal{R}$ is *weakly normalizing* (denoted by $\text{WN}(\mathcal{R})$) if for any S-expression s there exists an S-expression $t \in \text{NF}(\mathcal{R})$ such that $s \to_{\mathcal{R}}^* t$.

▶ **Lemma 5.10.** Let E be an elementary $\langle T, \Sigma \rangle$-theory and $\mathcal{R}$ be a left-linear simplicity-preserving STSRS on the same signature satisfying $\to_{\mathcal{R}}^{*} \subseteq \overset{\text{ext}}{\leftrightarrow}{}_{E}^{*}$, HQR($\mathcal{R}$) and WN($\mathcal{R}$). Then E is constructor-based.

Proof. Let $f \in \Sigma_{\text{d}}$ and σ_g be a substitution such that $\sigma_g : V^{\text{b}} \to \text{S}(\Sigma_{\text{c}})$. By WN($\mathcal{R}$), there exists $w \in \text{NF}(\mathcal{R})$ such that $(f\uparrow)\sigma_g \to_{\mathcal{R}}^{*} w$. Furthermore, since $(f\uparrow)\sigma_g \in \text{S}(\Sigma, V^{\text{f}})$, $w \in \text{S}(\Sigma, V^{\text{f}})$. By Lemma 5.4, $(f\uparrow)\sigma_g$ is simple. Since $\mathcal{R}$ is simplicity-preserving w is simple by Lemma 5.9. Thus by Lemma 5.5 and our assumption that $\mathcal{R}$ is left-linear and elementary and that HQR($\mathcal{R}$) holds, $w \in \text{S}(\Sigma_{\text{c}}, V^{\text{f}})$. By $\to_{\mathcal{R}}^{*} \subseteq \overset{\text{ext}}{\leftrightarrow}{}_{E}^{*}$, it follows that for any $f \in \Sigma_{\text{d}}$ and substitution $\sigma_g : V^{\text{b}} \to \text{S}(\Sigma_{\text{c}})$, there exists $w \in \text{S}(\Sigma_{\text{c}}, V^{\text{f}})$ such that $(f\uparrow)\sigma_g \overset{\text{ext}}{\leftrightarrow}{}_{E}^{*} w$. Thus E is constructor-based. ◀

▶ **Theorem 5.11** (checking natural inductive theorems)**.** Let $\mathcal{R}$ be a left-linear elementary natural simplicity-preserving STSRS such that WN($\mathcal{R}$) and HQR($\mathcal{R}$) hold. If $s\theta \overset{\text{ext}}{\leftrightarrow}{}_{\mathcal{R}}^{*} t\theta$ for any substitution $\theta : V^{\text{b}} \to \text{S}(\Sigma_{\text{c}})$, then $s \approx t$ is a natural inductive theorem of $\mathcal{R}$.

Proof. By Lemma 5.10, $\mathcal{R}$ is constructor-based. Then by Theorem 5.2 $s \approx t$ is a natural inductive theorem of $\mathcal{R}$. ◀

▶ **Example 5.12.** Let $\mathcal{R}$ be the STSRS given in Example 5.7. Then $\mathcal{R}$ is left-linear, elementary, simplicity-preserving and WN($\mathcal{R}$) and HQR($\mathcal{R}$) hold. To show that $\mathcal{R}$ is natural, we now show that there exists a natural $\langle \text{T}, \Sigma \rangle$-term algebra $\mathcal{A}$ for $\mathcal{R}$. Let $\mathsf{0}^{\mathcal{A}} = \mathsf{0}$, $\mathsf{s}^{\mathcal{A}}(x) = (\mathsf{s}\ x)$, $[\,]^{\mathcal{A}} = [\,]$, $:^{\mathcal{A}}(x, xs) = (:\ x\ xs)$, $+^{\mathcal{A}}(x, y)$ be the unique normal form of $(+\ x\ y)$, $\mathsf{app}^{\mathcal{A}}(xs, ys)$ be the unique normal form of $(\mathsf{app}\ xs\ ys)$ and $\mathsf{map}^{\mathcal{A}}(f, xs)$ be defined inductively as: $\mathsf{map}^{\mathcal{A}}(f, [\,]) = [\,]$; $\mathsf{map}^{\mathcal{A}}(f, : x\ xs) = (:\ f(x)\ \mathsf{map}^{\mathcal{A}}(f, xs))$. Then we have $[\![l]\!]_{\rho} = [\![r]\!]_{\rho}$ for any $l \to r \in \mathcal{R}$. Furthermore, one easily shows $s \overset{\text{ext}}{\leftrightarrow}{}_{\mathcal{R}}^{*} [\![s]\!]$ for any $s \in \text{S}(\Sigma)^{\text{Nat}}$ by induction on s. Using this, it also follows that $s \overset{\text{ext}}{\leftrightarrow}{}_{\mathcal{R}}^{*} [\![s]\!]$ for any $s \in \text{S}(\Sigma)^{\text{List}}$ by induction on s. Thus $\mathcal{A}$ is a natural $\langle \text{T}, \Sigma \rangle$-term algebra for $\mathcal{R}$. Let

$$l \approx r = \mathsf{map}\ F\ (\mathsf{app}\ xs\ ys) \approx \mathsf{app}\ (\mathsf{map}\ F\ xs)\ (\mathsf{map}\ F\ ys).$$

Then for any substitution $\theta : V^{\text{b}} \to \text{S}(\Sigma_{\text{c}})$, $l\theta \overset{\text{ext}}{\leftrightarrow}{}^{*}{}_{\mathcal{R}}\ r\theta$ holds. Thus, by Theorem 5.11, $l \approx r$ is a natural inductive theorem of $\mathcal{R}$.

6 Conclusion

Extensibility of inductive theorems is indispensable to extend the framework of program transformation by templates based on first-order term rewriting [7, 8, 9] to the higher-order setting. We have studied a new notion of inductive theorems for higher-order rewriting, *natural inductive theorems*, to incorporate properties such as extensionality and extensibility. The class of this theorems is placed between extensional theorems and extensional inductive theorems. We also have given sufficient conditions for natural inductive theorems which enables us to prove simply typed equations to be natural inductive theorems.

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