

## 2023年度 数理論理学 復習問題(14)

問題 1 以下の述語論理式の証明図を書け.

- (1)  $\forall x \forall y P(x, y) \rightarrow \exists z P(z, z)$
- (2)  $\forall x P(x, f(x)) \rightarrow \forall x \exists y P(x, y)$
- (3)  $\neg \exists x (\neg P(x) \wedge P(x))$
- (4)  $\forall x (P(x) \wedge Q(x)) \rightarrow (\exists x P(x) \wedge \exists x Q(x))$
- (5)  $\exists x (P(x) \rightarrow Q(x)) \rightarrow (\forall x P(x) \rightarrow \exists x Q(x))$
- (6)  $\exists x P(x) \vee \exists x Q(x) \rightarrow \exists x (P(x) \vee Q(x))$
- (7)  $\forall x (P(x) \rightarrow \exists y Q(x, y)) \rightarrow P(a) \rightarrow \exists y Q(a, y)$

問題 2 以下の述語論理式の証明図を書け.

- (1)  $\forall x \forall y \exists z (x + y \approx z)$
- (2)  $\forall x \forall y (P(x) \wedge x \approx y \rightarrow P(y))$
- (3)  $\forall x \exists y (x \approx f(y)) \rightarrow \forall x \exists y (x \approx f(f(y)))$

問題 3 以下の述語論理式の証明図を書け.

- (1)  $\forall x P(x) \wedge \forall y Q(y) \rightarrow \exists z (P(z) \wedge Q(z))$
- (2)  $\forall x P(x) \rightarrow \exists z P(z) \wedge \exists y P(y)$
- (3)  $\exists x P(x) \rightarrow \exists y \exists z (P(y) \wedge P(z))$
- (4)  $\exists x P(x) \rightarrow \forall x \forall y (P(x) \rightarrow Q(y)) \rightarrow \forall y Q(y)$
- (5)  $\exists x (Q \rightarrow P(x)) \rightarrow (Q \rightarrow \exists x P(x))$
- (6)  $\exists x P(x, x, x) \rightarrow \exists y \exists z P(y, y, z)$
- (7)  $\exists x \forall y P(x, x, y) \rightarrow \exists x \exists y P(x, y, y)$
- (8)  $\forall x \exists y (P(x) \rightarrow P(y))$
- (9)  $\forall x (P(x) \vee Q(x)) \rightarrow \exists x \neg P(x) \rightarrow \exists x Q(x)$
- (10)  $\exists x \neg P(x) \rightarrow \neg \forall x P(x)$
- (11)  $\neg \forall x P(x) \rightarrow \exists x \neg P(x)$

## 2023 年度 数理論理学 復習問題解答 (14)

### 問題 1

(1)

$$\frac{\frac{\frac{[\forall x \forall y P(x, y)]^1}{\forall y P(z, y)} \forall E}{\frac{P(z, z)}{\exists z P(z, z)} \exists I} \forall E}{\forall x \forall y P(x, y) \rightarrow \exists z P(z, z)} \rightarrow I^1$$

(2)

$$\frac{\frac{\frac{[\forall x P(x, f(x))]^1}{P(x, f(x))} \forall E}{\exists y P(x, y)} \exists I}{\forall x \exists y P(x, y)} \forall I}{\forall x P(x, f(x)) \rightarrow \forall x \exists y P(x, y)} \rightarrow I^1$$

(3)

$$\frac{\frac{[\exists x (P(x) \wedge \neg P(x))]^1}{\neg \exists x (\neg P(x) \wedge P(x))} \neg I^2}{\frac{\frac{[\exists x (P(x) \wedge \neg P(x))]^1}{\perp} \perp}{\exists x E^1} \perp}{\neg \exists x (\neg P(x) \wedge P(x))} \neg I^2}$$

(4)

$$\frac{\frac{\frac{[\forall x (P(x) \wedge Q(x))]^1}{P(x) \wedge Q(x)} \forall E}{\frac{P(x)}{\exists x P(x)} \exists I} \wedge E}{\frac{\frac{[\forall x (P(x) \wedge Q(x))]^1}{P(x) \wedge Q(x)} \forall E}{\frac{Q(x)}{\exists x Q(x)} \exists I} \wedge E} \wedge I}{\forall x (P(x) \wedge Q(x)) \rightarrow (\exists x P(x) \wedge \exists x Q(x))} \rightarrow I^1$$

(5)

$$\frac{\frac{\frac{[\exists x (P(x) \rightarrow Q(x))]^3}{\exists x Q(x)} \rightarrow I^2}{\forall x P(x) \rightarrow \exists x Q(x)} \rightarrow I^2}{\frac{[\exists x (P(x) \rightarrow Q(x))]^3}{\exists x Q(x)} \rightarrow I^2}{\frac{[\forall x P(x)]^2}{P(x)} \forall E}{\frac{P(x) \rightarrow Q(x)}{Q(x)} \rightarrow E} \rightarrow E} \rightarrow E}{\exists x (P(x) \rightarrow Q(x)) \rightarrow (\forall x P(x) \rightarrow \exists x Q(x))} \rightarrow I^3$$

(6)

$$\frac{\frac{[\exists x P(x) \vee \exists x Q(x)]^3}{\exists x (P(x) \vee Q(x))} \exists I}{\frac{\frac{[\exists x P(x)]^2}{\exists x (P(x) \vee Q(x))} \exists I}{\frac{P(x)}{P(x) \vee Q(x)} \forall I} \exists I} \exists E^1}{\frac{[\exists x Q(x)]^2}{\exists x (P(x) \vee Q(x))} \exists I}{\frac{Q(x)}{P(x) \vee Q(x)} \forall I} \exists I} \exists E^1}{\exists x (P(x) \vee Q(x))} \forall E^2}{\exists x P(x) \vee \exists x Q(x) \rightarrow \exists x (P(x) \vee Q(x))} \rightarrow I^3$$

(7)

$$\frac{\frac{\frac{[\forall x (P(x) \rightarrow \exists y Q(x, y))]^2}{P(a) \rightarrow \exists y Q(a, y)} \forall E \quad [P(a)]^1}{\exists y Q(a, y)} \rightarrow E}{\frac{P(a) \rightarrow \exists y Q(a, y)}{\rightarrow I^1}} \rightarrow I^2$$

## 問題 2

(1)

$$\frac{\frac{\frac{\frac{\overline{\forall x (x \approx x)}}{\forall x (x \approx x)} \text{REFL}}{x + y \approx x + y} \forall E}{\exists z (x + y \approx z)} \exists I}{\forall y \exists z (x + y \approx z)} \forall I}{\forall x \forall y \exists z (x + y \approx z)} \forall I$$

(2)

$$\frac{\frac{\frac{[P(x) \wedge x \approx y]^1}{x \approx y} \wedge E \quad \frac{[P(x) \wedge x \approx y]^1}{P(x)} \wedge E}{P(y)} \text{SUBST}}{\frac{P(x) \wedge x \approx y \rightarrow P(y)}{\rightarrow I^1}} \rightarrow I^1}{\frac{\forall y (P(x) \wedge x \approx y \rightarrow P(y))}{\forall I}} \forall I}{\forall x \forall y (P(x) \wedge x \approx y \rightarrow P(y))} \forall I$$

(3)

$$\frac{\frac{\frac{[\forall x \exists y (x \approx f(y))]^3}{\exists y (x \approx f(y))} \forall E \quad \frac{\frac{[\forall x \exists y (x \approx f(y))]^3}{\exists z (y \approx f(z))} \forall E \quad \frac{\frac{[y \approx f(z)]^1 \quad [x \approx f(y)]^2}{x \approx f(f(z))} \text{SUBST}}{\exists y (x \approx f(f(y)))} \exists I}{\exists y (x \approx f(f(y)))} \exists E^1}{\exists y (x \approx f(f(y)))} \exists E^2}{\frac{\exists y (x \approx f(f(y)))}{\forall x \exists y (x \approx f(f(y)))} \forall E}} \rightarrow I^3$$

## 問題 3

(1)

$$\frac{\frac{\frac{[\forall x P(x) \wedge \forall y Q(y)]^1}{\forall x P(x)} \wedge E \quad \frac{[\forall x P(x) \wedge \forall y Q(y)]^1}{\forall y Q(y)} \wedge E}{\frac{P(z)}{P(z)} \forall E \quad \frac{Q(z)}{Q(z)} \forall E}{P(z) \wedge Q(z)} \wedge I}{\frac{\exists z (P(z) \wedge Q(z))}{\exists I}} \exists I}{\forall x P(x) \wedge \forall y Q(y) \rightarrow \exists z (P(z) \wedge Q(z))} \rightarrow I^1$$

(2)

$$\frac{\frac{\frac{[\forall x P(x)]^1}{P(z)} \forall E \quad \frac{[\forall x P(x)]^1}{P(y)} \forall E}{\exists z P(z)} \exists I \quad \frac{\frac{[\forall x P(x)]^1}{P(y)} \forall E}{\exists y P(y)} \exists I}{\exists z P(z) \wedge \exists y P(y)} \wedge I}{\forall x P(x) \rightarrow \exists z P(z) \wedge \exists y P(y)} \rightarrow I^1$$

(3)

$$\frac{\frac{\frac{[P(y)]^1 \quad [P(y)]^1}{P(y) \wedge P(y)} \wedge I}{\exists z (P(y) \wedge P(z))} \exists I}{\frac{[\exists x P(x)]^2 \quad \exists y \exists z (P(y) \wedge P(z))}{\exists y \exists z (P(y) \wedge P(z))} \exists E^1} \exists E^1}{\exists x P(x) \rightarrow \exists y \exists z (P(y) \wedge P(z))} \rightarrow I^2$$

(5)

$$\frac{\frac{\frac{[Q \rightarrow P(x)]^1 \quad [Q]^2}{P(x)} \rightarrow E}{\exists x P(x)} \exists I}{\frac{[\exists x (Q \rightarrow P(x))]^3 \quad \exists x P(x)}{\exists x P(x)} \exists E^1} \exists E^1}{\frac{Q \rightarrow \exists x P(x)}{Q \rightarrow \exists x P(x)} \rightarrow I^2} \rightarrow I^2}{\exists x (Q \rightarrow P(x)) \rightarrow (Q \rightarrow \exists x P(x))} \rightarrow I^3$$

(7)

(4)

$$\frac{\frac{\frac{[\forall x \forall y (P(x) \rightarrow Q(y))]^2}{\forall y (P(z) \rightarrow Q(y))} \forall E}{P(z) \rightarrow Q(y)} \forall E}{\frac{[P(z)]^1}{P(z) \rightarrow Q(y)} \rightarrow E} \rightarrow E}{\frac{[\exists x P(x)]^3 \quad \frac{Q(y)}{\forall y Q(y)} \forall I}{\forall y Q(y)} \exists E^1} \exists E^1}{\frac{\forall x \forall y (P(x) \rightarrow Q(y)) \rightarrow \forall y Q(y)}{\forall x \forall y (P(x) \rightarrow Q(y)) \rightarrow \forall y Q(y)} \rightarrow I^2} \rightarrow I^2}{\exists x P(x) \rightarrow \forall x \forall y (P(x) \rightarrow Q(y)) \rightarrow \forall y Q(y)} \rightarrow I^3$$

(6)

$$\frac{\frac{[P(y, y, y)]^1}{\exists z P(y, y, z)} \exists I}{\frac{[\exists x P(x, x, x)]^2 \quad \exists y \exists z P(y, y, z)}{\exists y \exists z P(y, y, z)} \exists E^1} \exists E^1}{\exists x P(x, x, x) \rightarrow \exists y \exists z P(y, y, z)} \rightarrow I^2$$

(8)

(9)

$$\frac{\frac{\frac{[\forall y P(z, z, y)]^1}{P(z, z, z)} \forall E}{\exists y P(z, y, y)} \exists I}{\frac{[\exists x \forall y P(x, x, y)]^2 \quad \exists x \exists y P(x, y, y)}{\exists x \exists y P(x, y, y)} \exists E^1} \exists E^1}{\exists x \forall y P(x, x, y) \rightarrow \exists x \exists y P(x, y, y)} \rightarrow I^2$$

$$\frac{\frac{[P(x)]^1}{P(x) \rightarrow P(x)} \rightarrow I^1}{\frac{\exists y (P(x) \rightarrow P(y))}{\exists y (P(x) \rightarrow P(y))} \exists I} \exists I}{\forall x \exists y (P(x) \rightarrow P(y))} \forall I$$

(10)

$$\frac{\frac{\frac{[\forall x (P(x) \vee Q(x))]^4}{P(z) \vee Q(z)} \forall E}{\exists x Q(x)} \exists E^2}{\frac{[\exists x \neg P(x)]^3 \quad \exists x Q(x)}{\exists x Q(x)} \exists E^2} \exists E^2}{\frac{\exists x \neg P(x) \rightarrow \exists x Q(x)}{\exists x \neg P(x) \rightarrow \exists x Q(x)} \rightarrow I^3} \rightarrow I^3}{\forall x (P(x) \vee Q(x)) \rightarrow \exists x \neg P(x) \rightarrow \exists x Q(x)} \rightarrow I^4$$

(11)

$$\frac{\frac{\frac{[\exists x \neg P(x)]^3}{\neg \forall x P(x)} \exists E^2}{\exists x \neg P(x)} \exists E^2}{\frac{[\neg P(y)]^2 \quad \frac{\perp}{\neg \forall x P(x)} \neg I^1}{\neg \forall x P(x)} \neg E} \neg E}{\frac{[\forall x P(x)]^1}{P(y)} \forall E}{\frac{\perp}{\neg \forall x P(x)} \neg I^1} \neg I^1}{\exists x \neg P(x) \rightarrow \neg \forall x P(x)} \rightarrow I^3$$

$$\frac{\frac{\frac{[\neg \exists x \neg P(x)]^2 \quad \frac{[\neg P(x)]^1}{\exists x \neg P(x)} \exists I}{\perp} \neg E}{\frac{\perp}{P(x)} \text{RAA}^1} \neg E}{\frac{[\neg \forall x P(x)]^3 \quad \frac{\perp}{\forall x P(x)} \forall I}{\forall x P(x)} \neg E} \neg E}{\frac{\perp}{\exists x \neg P(x)} \text{RAA}^2} \neg E}{\neg \forall x P(x) \rightarrow \exists x \neg P(x)} \rightarrow I^3$$